Dynamic Efficiency and Asset Bubbles under Financial Market Friction [#]

Masaya Sakuragawa

masaya@econ.keio.ac.jp

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Abstract

This paper investigates whether the criterion for the real interest rate versus the economic growth rate is effective in evaluating dynamic efficiency/inefficiency and understanding asset bubbles in a dynamic economy with frictions of financial markets. In the closed economy, this criterion is effective for sustainability of bubbles, but not in terms of dynamic efficiency/inefficiency. Asset bubbles coexist with capital under-accumulation and do not necessarily improve dynamic efficiency. Asset bubbles emerge even when savings are so small that capital is less accumulated than the Golden Rule in the bubbleless economy. The effectiveness of that criterion is restored in the small-open economy if the output produced by borrowers is international collateral.

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1. Introduction

The criterion for the real interest rate versus the economic growth rate is a problem of theoretical and empirical importance in evaluating the efficiency of the economy and understanding sustainability of asset bubbles. Diamond (1965), Ihori (1978), and Tirole (1985) demonstrate that when the return to capital is equal to the real interest rate, this criterion becomes a benchmark for understanding dynamic efficiency and sustainability of bubbles. However, various kinds of financial frictions, including uncertainty, transaction costs, asymmetric information, and other incentive problems will deter this equality to continue to hold in the actual economy so that the effectiveness of this criterion is an open question. Abel et al (1989) and Bohn (1995) provide examples in which the safe interest rate is smaller than the economic growth rate due to risk premium but the economy is dynamically efficient.

This problem is stringent also from the empirical prospect because the current speculative bubbles will be associated with the historically low real interest rate. Figure 1 illustrates average interest rates and the average economic growth rate of G7 countries (U.S., U.K., Japan, Germany, France, Canada, and Italy).[1] Remarkably, before the end of 1990s, all the interest rates were greater than the economic growth rate except for the deposit rate, but after that period, many of the interest rates began to be lower than the economic growth rate. Since 2000 and around, several kinds of bubbles recurrently emerged, including IT bubbles, housing market booming in US, the booming in the global stock markets, and appreciations in gold and oil prices.[2]

[1] Each of interest rates and the economic growth rate is a simple average of G-7 countries (*source:* IFS). We exclude the 1991 data of German in calculating averages. Money market rate, treasury bills, treasury bills: 3years or longer, deposit rate, and max overdraft reflect call rate, short-term rate of the government bond, long-term rate of the government bond, short-term deposit interest rate, and loan interest, respectively.

[2] Additionally, behind asset bubbles in China that have been sustained for more than two decades is the far higher economic growth rate than the interest rates.

Tirole (1985) demonstrates that asset bubbles arise when capital is overly accumulated in the bubbleless economy. He argues the recurrent emergence of multiple bubbles using the terminology of "bubble substitution", but fails in explaining the coexistence of "global bubbles" and weak incentive for investment relative to great global savings, which is one of important features of the current world economy.[3] Caballero and Krishnamurthy (2006) develop an insightful model of a small-open economy in which capital under-accumulation and asset bubbles can coexist. In theirs, asset bubbles are used for collateral for financing productive investment. Their contribution is to show the complementary role of bubbles with capital accumulation, in contrast with Tirole where bubbles and capital accumulation are substitutable.

The purpose of this paper is to investigate the effectiveness of the criterion for the real interest rate versus the economic growth rate in evaluating dynamic efficiency/inefficiency and understanding asset bubbles in the presence of financial market frictions. In doing so, we construct a simple model which is as close as possible to Diamond (1965), Ihori (1978), and Tirole (1985) except for that there is a friction in the financial market. As the extensive literature argues, borrowing constraints or credit rationing that arises in response to incentive problems that prevail in financial markets leads to the breakdown of the equality between the return to capital and the real interest rate [e.g., Stiglitz and Weiss (1981), Gale and Hellwig (1985), Williamson (1986), Bernanke and Gertler (1989), and others]. The discrepancy between the two rates has different results on dynamic efficiency and sustainability of asset bubbles, relative to the standard economy.

In the closed economy, the criterion for the real interest rate versus economic growth rate is effective for sustainability of bubbles, but not in terms of dynamic efficiency/inefficiency. Asset bubbles move the capital stock down to the smaller level than the Golden Rule so that asset

[3] Caballero (2006) calls weak incentive for investment relative to great global savings the "asset shortage".

bubbles and capital under-accumulation coexist. Interestingly, asset bubbles emerge even when savings are so small that capital is less accumulated than the Golden Rule in the bubbleless economy. Asset bubbles may or may not restore dynamic efficiency, and do only if capital is far overly accumulated in the bubbleless economy. The theoretical finding explains asset bubbles, asset shortage, and the low real interest rate simultaneously, all of which features the current world economy. The effectiveness of that criterion is restored in the small-open economy with the real interest rate being constant if the output produced by entrepreneurs becomes international collateral. In the small-open economy, asset bubbles never crowd out capital accumulation, but has the role of raising the real interest rate faced by domestic investors.

We comment on related works. Abel et al (1989), Zilcha (1992), and Bohn (1995) demonstrate that, in a stochastic model, dynamic efficiency depends on the relation between the growth rate and the rate of return on "risky" capital, not on the one between the growth rate and the safe interest rate. Thus, the lower safe interest rate than the growth rate is consistent with dynamic efficiency and no-Ponzi condition.

Grossman and Yanagawa (1993) and Femminis (2002) provide endogenous growth models in which the presence of bubbles is welfare-reducing so that the condition for dynamic efficiency does not coincident with the one for sustainable bubbles. Ventura (2003) constructs a growth model of financial market friction in which bubbles can promote capital accumulation.This paper is organized as follows. Section 2 sets up the model and studies the benchmark economy. Section 3 analyzes the closed economy when there is the friction of financial market, and Section 4 the small-open economy. Section 5 concludes.

2. Basic Model

Let us consider an economy of overlapping generations that lasts for infinity. At each period $t = 0,1,2,\ldots,\infty,$ the economy is populated by a continuum of ex ante identical agents that live for two periods. Letting N_t denote the number of young people at t, the population grows at rate $n > 0$, satisfying $N_t = (1+n)^t N_0 = (1+n)^t$. At each period the final good is produced by firms that use labor and capital as inputs according to the constant-returns-to-scale technology described as $Y_t = F(K_t, N_t)$, where K_t and N_t are aggregate supplies of capital and labor, and Y_t is the output of the final good. That technology is described as a per-capita form by $y_t \equiv Y_t / N_t = F(K_t / N_t, 1) \equiv f(k_t)$, where k_t is the capital-labor ratio and y_t is the per-capita output of the final good. $f(.)$ is thrice continuously differentiable, increasing, concave, satisfying $f(0) = 0$, and $\lim_{k_t \to 0} f'(k_t) = +\infty$. Since the production technology is homogeneous of degree one, output of the final good can be described in terms of the action of a single, aggregate, price-taking firm. From the maximization problem of that firm, each input is paid its marginal product. The rate of return to capital R_t and the wage rate W_t are determined to satisfy $R_t = f'(k_t)$ and $W_t = f(k_t) - k_t f'(k_t) \equiv W(k_t)$, respectively. Assume that capital depreciates fully after one period. The final good is numeraire. The price of capital is equal to R_t. Each of ex ante identical agents born at t maximizes $\log c_t^y + bE_t \log c_{t+1}^o$, where $c_t^y (c_{t+1}^o)$ is consumption in the first (second) period of life, and E_t is the expectation operator. At the first period of life, each of them supplies one unit of time inelastically in the labor market Having received the wage repayment from the firm, each agent discovers his/her type. With probability a $(0 < a < 1)$, he is an “entrepreneur”, while with probability $1 - a$, he turns out to be an “investor”.

Each entrepreneur has access to one linear capital investment technology that transforms one unit of the final good into one unit of capital after one period. On the other hand, any investor cannot have access to the capital investment technology and earns the second-period income only by investing the first-period income to others.

Assume that there is no enforcement mechanism to fulfill financial contracts between debtors and creditors and hence to enforce on borrowers to repay their debt. When debtors breach the contract and refuse to make their repayment, a portion $l\,(0 < l < 1)$ of their earnings are assumed to be forfeited by the creditor. A low l is interpreted to capture weak bankruptcy procedure, poor bank monitoring, and low contract enforcement, and will be associated with poorly developed financial markets [e.g., La Porta et al (1997, 1998), Levine (1998), Levine et al (2000)].[4] The parameter l is thus interpreted to capture the efficiency of the broadly defined financial system.

First of all, we investigate an economy with perfect capital market that is characterized by perfect enforcement. Letting r_{t+1} denote the interest rate at t+1, any of entrepreneurs earns $f'(k_{t+1})i_t - (i_t - W_t)(1+r_{t+1})$ by investing an amount of i_t in his own project, while he earns $W_t(1+r_{t+1})$ by supplying his first-period income to others. Entrepreneurs are willing to start their own projects if

(2-1) $$f'(k_{t+1}) \geq 1 + r_{t+1}.$$

We call this inequality the *profitability constraint*. If $f'(k_{t+1}) > 1 + r_{t+1}$, any of them would demand the fund indefinitely as much as possible, which drives the interest rate up. Conversely, if $f'(k_{t+1}) < 1 + r_{t+1}$, any of them would stop the project, which drives the interest rate down. When agents borrow and lend indefinitely at the prevailing interest rate,

(2-2) $$f'(k_{t+1}) = 1 + r_{t+1}$$

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[4] La Porta et al (1997) find that countries with poor investor protections, measured by both the character of legal rules and the quality of law enforcement, have smaller capital markets. Levine (1998) shows that financial depth is closely linked with measures of legal treatment of outside creditors developed by La Porta et al(1997).

is only sustainable in equilibrium. Entrepreneurs and investors then earns $1+r_{t+1}$ per unit equally, and the uncertainty about his/her type is irrelevant.

Both entrepreneurs and investors save a fraction $s \equiv b/(1+b)$ of the first-period income. Letting B_t denote the aggregate bubbles, which can be best thought of pieces of paper, the aggregate saving is used for financing investment in capital and purchasing bubbles, and the relation is described by

(2-3) $$N_t sW(k_t) = a i_t N_t + B_t .$$

The definition of the aggregate capital is given by $K_{t+1} = a\, i_t N_t$. Letting b_t denote the aggregate bubbles per capita at t, (2-3) is expressed in per capita term by

(2-4) $$(1+n)k_{t+1} = sW(k_t) - b_t .$$

Under perfect foresight, bubbles have to earn the same rate of return as that on capital to satisfy $B_{t+1} = (1+r_{t+1})B_t$. The aggregate bubbles per capita grow to satisfy

(2-5) $$(1+n)b_{t+1} = (1+r_{t+1})b_t .$$

Lastly bubbles have to be non-negative;

(2-6) $$b_t \geq 0 .$$

We define two kinds of equilibria. *A bubbleless equilibrium* is defined as an equilibrium in which $b_t = 0$ for any t or b_t converges to zero if $b_t > 0$ for any t. *An asymptotically bubbly equilibrium* is defined as an equilibrium in which b_t does not converge to zero.[5]

First of all, we examine the analysis of steady states. The steady state of a bubbleless equilibrium is characterized by a pair $\{\bar{k}, \bar{r}\}$, satisfying $(1+n)\bar{k} = sW(\bar{k})$, and $1+\bar{r} = f'(\bar{k})$. On the other hand, the steady state of a bubbly asymptotically equilibrium is characterized by a pair $\{k_{GR}, r_{GR}, b_{GR}\}$, satisfying $(1+n)k_{GR} = sW(k_{GR}) - b_{GR}$, $1+r_{GR} = f'(k_{GR})$, and $r_{GR} = n$,

[5] Note that Tirole (1985) distinguishes between an *asymptotically bubbly equilibrium* and a *bubbly equilibrium* by defining the latter as the one in which $b_t > 0$ for any t.

with $b_{GR} > 0$.

We briefly summarize the properties of the economy under perfect financial markets. If $\bar{r} > n$, there exists a unique bubbleless equilibrium and the interest rate converges to $\bar{r}$, while otherwise, there exists an asymptotically bubbly equilibrium and the interest rate converges to n [Tirole (1985, Proposition 1)]. Ihori (1978) demonstrates that the government bond, which is intrinsically valueless, carries the long-run capital level to the Golden Rule level of capital when $\bar{r} < n$.

3. The Economy with Financial Market Imperfection

We now introduce the imperfection of financial markets into the benchmark model. The financial market is competitive in the sense that both entrepreneurs (borrowers) and investors (lenders) take the equilibrium rate r_{t+1} as given. Let x_t denote the amount of saving that is used for internal wealth added for the investment project. If any of entrepreneurs borrows $(i_t - x_t)$ and repays $(1 + r_{t+1})\ (i_t - x_t)$ to investors honestly, he earns $f'(k_{t+1})i_t - (i_t - x_t)(1 + r_{t+1})$, while if he breaches the promise for repayment, a portion l of his earning is forfeited, and his earning would be $(1 - l)\, f'(k_{t+1})i_t$. The following incentive compatibility constraint summarizes the imperfection of the financial markets, i.e.,

(3-1) $$(i_t - x_t) \times (1 + r_{t+1}) \leq l f'(k_{t+1}) i_t .$$

Equation (3-1) will be called the "*borrowing constraint*". Entrepreneurs can borrow the amount up to some fraction of the project revenue.[6] One may derive the similar borrowing constraint from a number of other incentive considerations.[7] When the borrowing constraint binds with

[6] Implicit in (3-1) is that entrepreneurs do not use the borrowed fund to buy bubbles. Entrepreneurs will borrow only for capital investment because the rate of return from capital is greater than the one from holding bubbles when (3-1) binds with equality, as argued below.

[7] For example, it is possible to derive the borrowing constraint by engaging in the costly-state-verification approach that began with Townsend (1979), and has been developed by Gale

equality, the rate of return faced by entrepreneurs is $z(r_{t+1}, k_{t+!}) \equiv (1-l)\dfrac{f'(k_{t+1})(1+r_{t+1})}{1+r_{t+1}-lf'(k_{t+1})}$,

which is assumed to be positive at this stage. Each agent that saves the amount of x_t earns $x_t z(r_{t+1}, k_{t+1})$ if he/she is an entrepreneur, while $x_t(1+r_{t+1})$ if he/she is an investor. Ex ante identical agent turns out to choose the amount of saving x_t to maximize

$$\log(W_t - x_t) + b \log x_t + b(1-a)\log(1+r_{t+1}) + ba \log z(r_{t+1}, k_{t+1}) .$$

Obviously the agent chooses $x_t = sW_t$ since the agent's saving behavior is independent of the return to saving. It is useful to consider first the economy in which there are no bubbles. The market clearing in the capital market is given by

(3-2) $$a(i_t - sW(k_t)) = (1-a)sW(k_t) ,$$

Where the L.H.S. is the demand for funds by entrepreneurs, and the R.H.S. is the supply of fund by investors.

We turn to the determination of the real interest rate. At least either of the two constraints, the profitability constraint and the borrowing constraint, should bind with equality. If the borrowing constraint is not binding, the profitability constraint should bind with equality, while if the borrowing constraint is binding with equality, the profitability constraint may not be binding. The above argument is summarized by

(3-3) $$1+r_{t+1} = \min\{ f'(k_{t+1}), lf'(k_{t+1})\frac{i_t}{i_t - x_t}\} .$$

Without loss of generality, we confine attention on symmetric equilibria in which all entrepreneurs choose the same amount of investment. Equation (3-3), using (3-2), reduces to

(3-4) $$1+r_{t+1} = \min\{ f'(k_{t+1}), \frac{l}{1-a} f'(k_{t+1})\} .$$

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and Hellwig (1985) and Williamson (1986).In addition, if banks are established as an optimal response to the CSV problem [e.g. Diamond (1984)], r_{t+1} will be interpreted as the "safe" deposit interest rate, to be distinguished from the risky loan interest rate.

It is straightforward to see that the borrowing constraint is binding if and only if $a + l < 1$, and then we have $1 + r_{t+1} = \frac{l f'(k_{t+1})}{1-a}$. Hereafter we impose the following in order to focus on an interesting case.

Assumption 1 $a + l < 1$.

The fraction of entrepreneurs a is a measure of separation between creditors and debtors, and matters when the borrowing constraint is crucial. As $a \to 1$, outside funds are negligible, and all investment is carried out directly by entrepreneurs, while as $a \to 0$, outside funds are more important, and each of entrepreneurs has to borrow the greater amount from investors. Another parameter l capture the development of the the contract enforcement mechanism as argued above. As $l \to 1$, the incentive compatibility constraint is always satisfied, and entrepreneurs would be able to borrow as much as possible, taking r_{t+1} as given. As $l \to 0$, entrepreneurs would be able to borrow nothing and hence have to self-finance their investment entirely. As either a declines or l rises, Assumption 1 is more likely to be satisfied. Assumption 1 is intended to describe an economy with the severe borrowing constraint. Note that when Assumption 1 is satisfied, we have $1 + r_{t+1} = \frac{l f'(k_{t+1})}{1-a} > l f'(k_{t+1})$, which guarantees $z(r_{t+1}, k_{t+1})$ to be positive.

The steady state of an economy with the binding borrowing constraint is characterized by the pair $\{\tilde{k}, \tilde{r}\}$, satisfying

(3-5) $$\tilde{k} = \frac{s}{1+n} W(\tilde{k}),$$ and

(3-6) $$1 + \tilde{r} = \frac{l}{1-a} f'(\tilde{k}).$$

It is straightforward to see that the levels of the steady state capital stock and marginal products of capital are the same between the two economies with and without the borrowing constraint ($\tilde{k} = \bar{k}$). The real interest rate differs between the two economies. It follows from $1+\bar{r} = f'(\bar{k})$, (3-6), $\tilde{k} = \bar{k}$, and Assumption 1 that the binding borrowing constraint is a source of the declining interest rate. The possible shortage of demand for investment that arises from the borrowing constraint has to be adjusted by decline in the interest rate to equal the predetermined aggregate savings.[8] In other words, the marginal product of capital diverges from the equilibrium interest rate when the borrowing constraint binds with equality.

We now turn to the analysis of the economy with bubbles when there is an imperfection of financial markets. The market clearing in the credit market is rewritten as

(3-7) $$a(i_t - sW(k_t)) = (1-a)sW(k_t) - b_t .$$

When the borrowing constraint binds with equality, (3-1) is finally expressed by

(3-8) $$\{(1-a)sW(k_t) - b_t\}(1+r_{t+1}) = lf'(k_{t+1})\{sW(k_t) - b_t\} .$$

Equations (2-4), (2-5), (2-6), and (3-8) define an asymptotically bubbly equilibrium with $b_t > 0$ for $t \to \infty$. The steady state of an asymptotically bubbly equilibrium, if it exists, is characterized by a pair $\{\tilde{k}_B, \tilde{b}, \tilde{r}_B\}$, satisfying

(3-9) $$\tilde{k}_B = \frac{1}{1+n}\{W(\tilde{k}_B) - \tilde{b}\}$$

(3-10) $$\{(1-a)sW(\tilde{k}_B) - \tilde{b}\}(1+\tilde{r}_B) = lf'(\tilde{k}_B)\{sW(\tilde{k}_B) - \tilde{b}\},$$

(3-11) $$\tilde{r}_B = n, \text{ and}$$

(3-12) $$\tilde{b} > 0 .$$

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8 A number of overlapping generations models, including Azariadis and Smith (1993), Sakuragawa and Hamada (2001), Matsuyama (2004), and others, have shown that the introduction of the borrowing constraint, regardless of whether it is either endogenously derived as a response to the incentive problem or exogenously imposed, leads to a decline in the equilibrium interest rate.

We investigate general features of the bubbly equilibrium. Allowing for non-binding borrowing constraint, (3-1) is rewritten, using (3-7) and $(1+n)k_{t+1} = ai_t$, as

(3-13) $$1 + r_{t+1} \le lf'(k_{t+1}) \frac{(1+n)k_{t+1}}{(1+n)k_{t+1} - asW(k_t)} .$$

Following (3-3), the real interest rate is thus determined to satisfy

(3-14)) $$1 + r_{t+1} = \min\{ f'(k_{t+1}),\ lf'(k_{t+1}) \frac{(1+n)k_{t+1}}{(1+n)k_{t+1} - asW(k_t)} \}$$

Given this preparation, we obtain the following.

Proposition 1

If the borrowing constraint is binding with equality, the asymptotically bubbly equilibrium never achieves the Golden Rule, and attains capital under-accumulation.

Proof. The steady-state relation between k and r is expressed as

(3-15) $$1 + r = \min\{ f'(k),\ lf'(k) \frac{(1+n)k}{(1+n)k - asW(k)} \} \equiv \min\{ f'(k), \Lambda(k) \}$$

In the (k, r) plane, the real interest rate should be lower between the two curves. In Figure 2, $\Lambda(k)$ is illustrated to be always lower than $f'(k)$. When the borrowing constraint is binding, for any given r, the steady state level of capital stock has to be smaller than the one in the benchmark economy, and hence the one in the bubbly equilibrium, $\tilde{k}_B$, should be smaller than k_{GR}. Q.E.D.

Asset bubbles never achieve the Golden Rule in the presence of the friction of financial markets, but rather move the capital stock to the smaller level than the Golden Rule level, eventually giving rise to capital under-accumulation. Remarkably, asset bubbles are coexistent with capital under-accumulation, which is sharply contrasted with Diamond (1965), Ihori (1978), and Tirole (1985). The theoretical finding explains the coexistence of asset bubbles, asset shortage, and the low real interest, which is a striking feature of the current world economy. The space for parameter values under which the bubbly equilibrium exists is easily checked. As the saving rate s increases, $\tilde{k}$ is high {from (3-5)}and $f'(\tilde{k})$ is low, and hence $\tilde{r}$ tends to be low {from (3-6)}. Taking $f'(\tilde{k})$ as given, as either l or a is small, $\tilde{r}$ is more likely to be smaller than n. Other things being equal, a higher s, a smaller l or a is likely to leads to the emergence of the bubbly equilibrium.

Demonstrating the dynamic properties is almost similar to Tirole (1985) and Weil (1987) except for the derivation of the locus $b_{t+1} = b_t$. The curve $b_{t+1} = b_t$ follows from (2-5) and (3-7), and is given by

$$(1+n)\{(1-a)sW(k_t) - b_t\} = l f'(\frac{sW(k_t)-b_t}{1+n})\{sW(k_t)-b_t\}. \quad (3\text{-}16)$$

There exists a continuously differentiable function $b_t = \Phi(k_t)$, satisfying (3-16), with the derivative;

$$\Phi'(k_t) = \frac{-l f'(k_{t+1})/(1+n) + as(1+n)b_t W'(k_t)/\{sW(k_t)-b\}^2}{-l f''(k_{t+1})/(1+n) + s(1+n)aW(k_t)/\{sW(k_t)-b_t\}^2} > 0. \quad (3\text{-}17)$$

So long as the curve $b_{t+1} = b_t$ crosses the curve $k_{t+1} = k_t$ from below, dynamic properties are qualitatively the same as those developed by Tirole (1985) and Weil (1987). The properties of equilibria are summarized as follows.

Proposition 2

(a) If $\tilde{r} > n$, there exists a unique equilibrium. This equilibrium is bubbleless and the interest rate converges to $\tilde{r}$.

(b) If $f'(\tilde{k}) > 1+n > 1+\tilde{r}$, there exists a unique asymptotically bubbly equilibrium with initial bubble b_0. The per-capita bubble converges to $\tilde{b}$ and the interest rate converges to n.

In the asymptotically bubbly equilibrium, the steady-state per-capita capital, denoted $\tilde{k}_B$,

satisfies $f'(\tilde{k}_B) > 1+n = f'(k_{GR})$, with $\tilde{k}_B < \tilde{k} < k_{GR}$.

(c) If $1+n > f'(\tilde{k}) > 1+\tilde{r}$, there exists a unique asymptotically bubbly equilibrium with initial bubble b_0. The per-capita bubble converges to $\tilde{b}$ and the interest rate converges to n. In the asymptotically bubbly equilibrium, the steady-state per-capita capital satisfies $f'(\tilde{k}_B) > 1+n = f'(k_{GR})$, with $\tilde{k}_B < k_{GR} < \tilde{k}$.

A heuristic proof of Proposition 2 is as follows. Agents require that, at the stationary state, the rate of return on bubbles, $1+n$, be at least equal to the rate of return on lending, $1+\tilde{r}_B$, so that it must be the case that $n \geq \tilde{r}_B$ if $\tilde{b} > 0$. On the other hand, at the stationary state, the presence of bubbles decreases the capital stock relative to the bubbleless equilibrium, so that we must have $\tilde{k}_B < \tilde{k}$ and hence $\tilde{r}_B > \tilde{r}$ if $\tilde{b} > 0$. Therefore, the necessary condition for bubbles to be sustainable is $n > \tilde{r}$. Conversely, if $n > \tilde{r}$, bubbles absorb the aggregate savings and reduces the capital stock until the interest rate $1+\tilde{r}$ is pushed up to $1+n$. Finally, if $\tilde{r} > n$, it must be the case that $\tilde{r}_B = \frac{l}{1-a} f'(\tilde{k}_B) - 1 > \tilde{r}$, but then it follows from (2-5) that the aggregate bubbles per capita should grow indefinitely, which is infeasible.

Asset bubbles are sustainable if and only if $\tilde{r} < n$. The return to capital $f'(\tilde{k})$ is irrelevant to the criterion for the viability of bubbles. In contrast to Tirole (1985), asset bubbles are sustainable even when capital is less accumulated than the Golden Rule in the bubbleless economy is less (Proposition 2(b)). Figure 3 represents one typical configuration of the equilibrium dynamics when the steady state capital stock of the bubbleless equilibrium is smaller than the one of the Golden Rule, such that $\tilde{k}_B < \tilde{k} < k_{GR}$. The asymptotically bubbly equilibrium W is a global saddlepoint. All dynamic paths originating from below b_0 converges toward the bubbleless steady state D. Trajectories starting above b_0 are infeasible as they all lead to the resource constraint being violated in finite periods.

We see that if there is a friction in financial markets, asset bubbles emerge for greater parameter space than otherwise. We obtain the following

Corollary

Asset bubbles are sustainable for the greater parameter space of the saving rate s in the economy with financial market friction than without it. The smaller is either a or l, the parameter space of the admissible saving rate is even greater.

Proof: Let $\tilde{s}$ denote the saving rate that satisfies $1+n = \frac{l}{1-a} f'(k(\tilde{s}, n))$, and let $\bar{s}$ denote the saving rate that satisfies $1+n = f'(k(\bar{s}, n))$, where $k = k(s,n)$ is implicitly derived from (3-5), and increasing in s. Under Assumption 1, $\tilde{s} < \bar{s}$ holds. Furthermore $\tilde{s}$ is decreasing if either a or l decreases. Q.E.D.

The high saving rate is not necessary for asset bubbles to emerge if there is a friction in financial markets. To the extent that the borrowing constraint is severe, asset bubbles are more likely to arise even for the small saving rate.

I now turn to the question of efficiency.[9] The argument is complicated by the finding that

[9] If an intra-generational transfer of income would be permitted between investors and entrepreneurs through government intervention at the first period of their lives, an appropriate tax-subsidy scheme will move the economy substantially to the Diamond-Tirole model. The bubbleless equilibrium is dynamically efficient if $\tilde{r} > n$, while otherwise, the bubbleless equilibria are dynamically inefficient and the asymptotically bubbly equilibrium is dynamically efficient [Tirole (1985, Proposition 2)].

the asymptotically bubbly equilibrium realizes the smaller per-capita consumption than the Golden Rule (Proposition 1). If $\tilde{r} > n$, the efficiency result is straightforward because the sustainable bubbles are ruled out under perfect foresight.

If $\tilde{r} < n$, we have interesting findings. If $f'(\tilde{k}) > 1+n > 1+\tilde{r}$, in the original steady state of the bubbleless equilibrium, the per-capita consumption is less than the Golden Rule. The introduction of bubbles moves the capital stock down to the level smaller than the Golden Rule level, and decreases the per-capita consumption.

Finally, we consider the case for $1+n > f'(\tilde{k}) > 1+\tilde{r}$ in which capital is overly accumulated in the bubbleless equilibrium. In Figure 4, there exists a $\underline{k}$, under which the per-capita aggregate consumption is less than k_{GR} and the same as the one in $\tilde{k}$, satisfying $f(\underline{k}) - (1+n)\underline{k} = f(\tilde{k}) - (1+n)\tilde{k}$. The welfare implications differ according to whether the steady state of the asymptotically bubbly equilibrium $\tilde{k}_B$ lies greater than $\underline{k}$ or not. We summarize the following.

Proposition 3

(a) If $\tilde{r} > n$, the bubbleless equilibrium is dynamically efficient.

(b) If $f'(\tilde{k}) > 1+n > 1+\tilde{r}$, the asymptotically bubbly equilibrium does not improve efficiency. The bubbleless equilibrium is dynamically efficient.

(c) If $1+n > f'(\tilde{k}) > 1+\tilde{r}$, the asymptotically bubbly equilibrium may or may not improve efficiency, depending on parameter values. If $\tilde{k}_B < \underline{k}$, the asymptotically bubbly equilibrium does not improve efficiency, and the bubbleless equilibrium is dynamically efficient, while if

$\tilde{k}_B > \underline{k}$, it improves efficiency so that the bubbleless equilibrium is dynamically inefficient.[10]

As Proposition (b) and (c) state, the inequality $n > \tilde{r}$ is not sufficient for the bubbleless equilibrium to be dynamically inefficient so that asset bubbles arise even when the bubbleless economy is dynamically efficient. Additionally, as Proposition (c) states, $1+n > f'(\tilde{k})$ is not sufficient for the bubbleless equilibrium to be dynamically inefficient. In other words, the greater level of capital than the Golden rule is possible under the dynamically efficient economy. Asset bubbles can restore efficiency only when capital is far overly accumulated. This is easily checked by looking at Figure 4. As $\tilde{k}$ increases, $\underline{k}$ decreases, and $\tilde{k}_B$ is more likely to be greater than $\underline{k}$. In sharp contrast with Tirole (1985), neither $n > \tilde{r}$ nor $1+n > f'(\tilde{k})$ is not a criterion under which asset bubbles restore dynamic efficiency.

4. Introduction of a Foreign Asset

Having thus far studied the closed economy, we extend the model by assuming that investors have access to a foreign asset with a constant interest rate r in order to investigate the small-open economy. . This experiment allows us to understand to what extent general features derived in the closed economy continue to hold in the small-open economy. We impose two further assumptions. First, domestic agents perceive bubbles to continue in the next period in the bubbly equilibrium, but foreign people do not. This assumption allows domestic agents only to hold bubbles. Second, the capital good is used as "international collateral" [e.g., Caballero and Krishnamurthy, 2001] and foreign entrepreneurs can pledge the capital good as collateral. The latter assumption allows domestic entrepreneurs to borrow from foreign investors.

[10] The proof of © is left to Appendix.

As argued below, pledgeability is essential to derive main results in this section. The small-open economy version is a simplified one of Caballero and Krishnamurthy (2006) in two substantial ways. In theirs, bubbles are stochastic and used as collateral for financing productive investment.

We first investigate the bubbleless economy. Since entrepreneurs find the cost of fund as r, the borrowing constraint is described as

(4-1) $$(i_t - x_t)\times(1+r) \le l f'(k_{t+1}) i_t \text{.}^{11}$$

Equation (4-1) is rewritten, using $(1+n)k_{t+1} = a i_t$, as

(4-2) $$\left(\frac{1+n}{a}k_{t+1} - sW_t\right)\times(1+r) \le l f'(k_{t+1})\frac{1+n}{a}k_{t+1}.$$

The borrowing constraint is binding with equality when two inequalities, $(1+n)k_{t+1} > asW_t$ and $1+r > l f'(k_{t+1})$, are satisfied. If either of the two is not met, the equilibrium is not borrowing constrained. Thus without loss of generality, we focus on an economy with $k_t > \underline{k} = \max\{\widehat{k}, \breve{k}\}$, where $(1+n)\widehat{k} = asW(\widehat{k})$ and $1+r = l f'(\breve{k})$. Given $k_t > \underline{k}$, (4-2) determines the evolution of k_t. From the application of the implicit function theorem, there exists a continuously differentiable function

(4-3) $$k_{t+1} = \Omega(k_t),$$

with $\Omega'(.) = \dfrac{as(1+r)W'(k_t)}{(1+n)\{1+r-l f'(k_{t+1}) - l f''(k_{t+1})\}} > 0$ for $k_t \in (\underline{k}, +\infty)$. Given k_0, k_t converges to the steady state level of the capital stock, denoted $k(r)$, with $k(r) = \Omega(k(r))$.[12] We easily find that $k(r) > \tilde{k}$ if and only if $r < \tilde{r}$. The equilibrium involves the capital inflow

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[11] Letting $\tilde{r}$ denoting the domestic interest rate, if $\tilde{r} < r$, domestic investors find it more beneficial to invest abroad, and the domestic interest rate rises until it is equal to the world interest rate r, while $\tilde{r} > r$, entrepreneurs find it more beneficial to fund from foreign investors, and the domestic interest rate falls until it is equal to r.

[12] There may possibly exist multiple $k(r)$'s, but we do not refer to that possibility because the multiplicity of equilibria is out of focus in this paper.

or outflow. The capital inflow of $(1+n)k_{t+1} - sW_t$ is positive (negative) if $r < \tilde{r}$ ($r > \tilde{r}$).

We next investigate the asymptotically bubbly equilibrium. Bubbles evolve as

(4-4) $$b_{t+1} = \frac{1+r_{t+1}^B}{1+n} b_t .$$

As argued below, entrepreneurs have no incentive to hold bubbles, and so only domestic investors hold bubbles.,

(4-5) $$(1-a)sW_t = b_t ,$$

if $r_{t+1}^B > r$, or $(1-a)sW_t \geq b_t$ if $r_{t+1}^B = r$. Note that as will be made clear below, r_{t+1}^B may be higher than r.

We primarily investigate an interesting case of $n > r$, the case of which corresponds to the one in which bubbles are viable in the closed economy. Since then the return from bubbles is greater than the one from lending at least near the steady state, there exists an equilibrium with sustainable bubbles.

We distinguish between two cases of (i) $1+n > 1+r > 1+\tilde{r}$ and (ii) $1+n > 1+\tilde{r} > 1+r$. Consider first the case for $1+n > 1+r > 1+\tilde{r}$. Equations (4-3), (4-4), and (4-5) define a (asymptotically) bubbly equilibrium with $b_t > 0$ for $t \to \infty$. Equation (4-5) is implied by the conjecture that the rate of return of bubbles is higher than that of lending to entrepreneurs at least near the steady state. The determination of equilibrium is recursive. Equation (4-3) determines the evolution of k_t. Given k_t, (4-4) determines b_t. Finally, the sequence of b_t determines r_{t+1}^B. The steady state is characterized by a pair $\{k^B(r), b(r), r^B\}$, satisfying $k^B(r) = \Omega(k^B(r))$, $r^B = n$, and $(1-a)sW(k^B(r)) = b$. Domestic investors hold bubbles and never lend to entrepreneurs. Entrepreneurs borrow entirely from foreign investors. Entrepreneurs never hold bubbles by borrowing because bubbles are not effective as collateral for foreign borrowing. This argument supports the equilibrium.

This economy enjoys the capital inflow of $(1+n)k_{t+1} - asW_t (>0)$. Bubbles lead to the reversal of capital flows from "out" to "in". Despite the presence of bubbles, the level of capital remains unchanged $(k(r) = k^B(r))$. It turns out that the same amount as channeled from domestic savings into bubbles is compensated for by the inflow of capital.

We now turn to the efficiency problem. The per-capita consumption is written as

(4-6) $C = \{ f(k(r)) - (1+n)k(r)\} + (r-n) \times outflow$.[13]

The first term remains unchanged before and after the introduction of bubbles. The second term represents the "net" inflow of capital. In an economy with $r < n$, reducing capital outflows or the reversing the direction of capital flow from "out" to "in" improves efficiency. The bubbleless economy is dynamically efficient in the sense that bubbles can reduce capital outflows, and thus improve efficiency dynamically.

In the case for $1+n > 1+\tilde{r} > 1+r$, all results continue to hold except for one point. In the bubbleless economy also, the economy enjoys the inflow of capital. By the presence of bubbles, the economy faces the larger amount of capital inflows. Of course, efficiency result is preserved. Note that whether $f'(k)$ is greater than $1+n$ or not is irrelevant to general features because k is entirely determined by parameters in this small-open economy. Consider finally the case of $n < r$. The efficiency result is trivial because bubbles are ruled out under perfect foresight. We summarize the following.

Proposition 4

In the small open economy, there exists an equilibrium with sustainable bubbles if and only if $r < n$. Furthermore, bubbles improve efficiency dynamically and hence the bubbleless

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[13] Strictly expressing, we have $C_t = f(k_t) - (1+n)k_{t+1} + (1+r)outflow_t - (1+n)outflow_{t+1}$, but the steady state expression is enough for the welfare analysis because $k_t = k(r)$ continues to hold.

economy is dynamically inefficient if $r < n$.

The efficiency result is quite different from the one in the closed economy. Even in the presence of financial market friction, the smaller real interest rate than the economic growth rate is a necessary and sufficient condition under which the bubbleless economy is dynamically inefficient, and the presence of bubbles improves efficiency. The efficiency result is coincident with Caballero and Krishnamurthy (2006).

Finally note that this finding is dependent entirely on the assumption that the capital good becomes international collateral. If otherwise, we find quite different arguments. The presence of bubbles crowds out domestic capital formation because entrepreneurs now can not rely on foreign borrowing for funding their projects. Entrepreneurs have to offer at least the same rate of return as the one of bubbles so that the economy attains the same allocation as the closed economy. The direction of efficiency is ambiguous. In (4-6), the economy incurs the loss from the first term because bubbles strengthen capital under-accumulation, but obtains the gain from the second term because capital outflows become zero.

Conclusion

In this paper we studied whether the criterion for the real-interest-rate versus the-economic-growth-rate is effective in evaluating dynamic efficiency/inefficiency and understanding rational bubbles in a dynamic economy with frictions of financial markets. We find that it is effective in the small-open economy, but not in the closed economy. Particularly, we find that when the real interest rate is endogenously determined in the model, the criterion for sustainable bubbles and the one for dynamic efficiency differ, and that difference generates several interesting features that are not found in the standard economy of Diamond (1965), Ihori

(1978), and Tirole (1985).

This study may be a small step to study global bubbles and the historically low real interest rate, and furthermore "global imbalance" in the world economy.[14] To answer whether the global bubbles make the world economy more efficient is also an important question. Theoretical findings in this paper suggest that the small open economy is better off by creating bubbles while the closed economy may not. The conflicting results on efficiency suggest that the conclusion depends on what kinds of role of bubbles we will focus on. In this paper we focus on the two aspects of bubbles, one is the role of raising the real interest rate, and the other of crowding out capital accumulation. As pointed out by Caballero and Krishnamurthy (2006), if the role of bubbles as collateral for financing productive investment is also focused on, we may have different results.

The third role of bubbles may be one key element to answer a question on whether asset bubbles and capital accumulation are substitutable or complementary. Given the resource constraint, asset bubbles will crowd out capital accumulation, on one hand, but on the other hand, if asset bubbles are used as collateral for borrowing, they will promote capital accumulation. The more enriched model will provide a foundation that helps us to understand bubbles in the presence of financial market frictions and hence the current world economy.

Appendix

The proof of the former part of © is similar to that of (b), and is omitted. I make the proof of the latter. By the introduction of bubbles, k_t monotonically decline from $\tilde{k}$ to $\tilde{k}_B$.

Letting C_T denote the per-capita aggregate consumption at period T at

[14] Caballero (2006) suggests the approach of the financial-market-friction view to be promising to solve "global imbalance".

$k_T = \tilde{k}$, $C_T(1+n)^{-1} = f(\tilde{k}) - (1+n)\tilde{k}$. The per-capita aggregate consumption at period at period $T+1$ becomes $C_{T+1}(1+n)^{-1} = f(\tilde{k}) - (1+g)k_{T+1} > C_T(1+n)^{-1}$ so long as $k_{GR} < k_{T+1} < \tilde{k}$. Furthermore, $C_{T+2}(1+n)^{-1} = f(k_{T+1}) - (1+n)k_{T+2} > f(k_{T+1}) - (1+n)k_{T+1}$ $> f(\tilde{k}) - (1+n)\tilde{k} = C_T(1+n)^{-1}$ if $k_{GR} < k_{T+2} < k_{T+1} < \tilde{k}$. This process continues so long as $k_{T+N} > k_{GR}$ for $N = 1,2,....$.

The introduction of bubbles makes the equilibrium path of capital monotonically decline. There exists some period $T+\hat{N}$ when $k_{T+\hat{N}}$ is less than k_{GR}. For any k_S, satisfying $\underline{k} < \tilde{k}_B < \ldots < k_{S+1} < k_S < \ldots < k_{GR}$, the followings are obtained; $\underline{C}(1+n)^{-1} = f(\underline{k}) - (1+n)\underline{k}$ $< f(k_S) - (1+n)k_S < f(k_S) - (1+n)k_{S+1} = C_S(1+n)^{-1}$.Over the whole process from the bubbleless steady state toward the new steady state, the aggregate consumption is greater than the one in the bubbleless steady state so long as $\tilde{k}_B > \underline{k}$. Q.E.D.

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Figure 2

Capital under-Accumulation and Borrowing Constraint

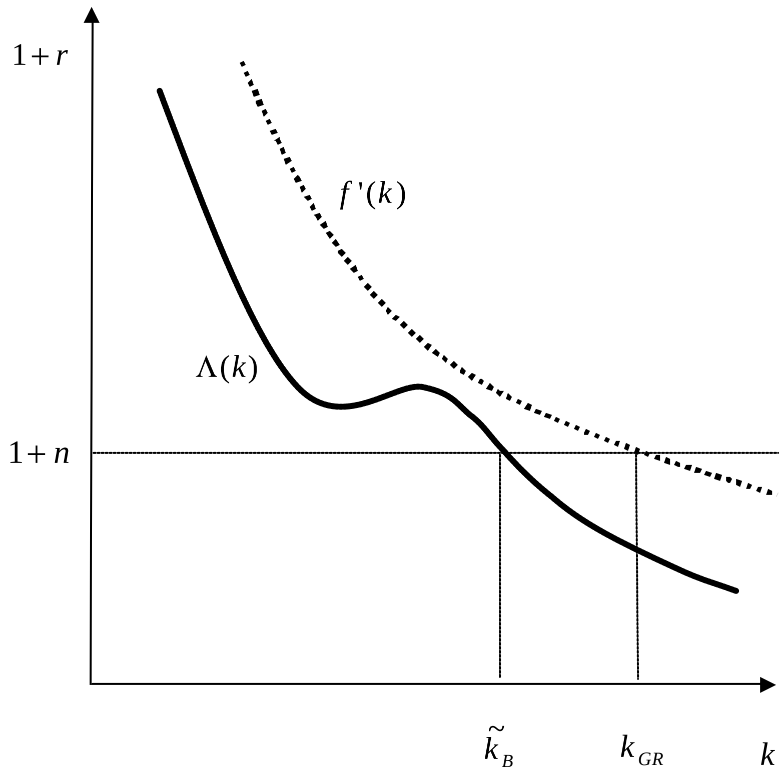

Figure 3 **Equilibrium Dynamics**

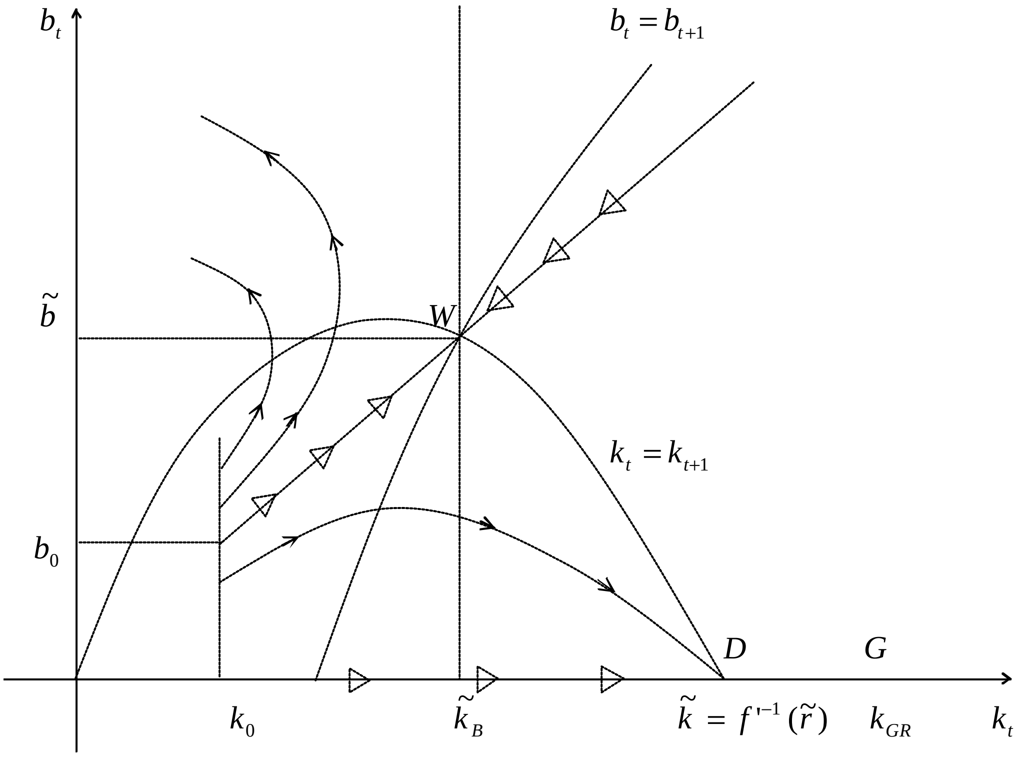

Figure 4 **Per-capita Consumption and Dynamic Efficiency**