ADVANCED MACROECONOMICS: FINAL EXAMINATION

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Consider an optimal growth model with an infinite horizon in which a representative agent solves the following maximization problem:

$$\max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \ln c_0 + \beta \ln c_1 + \beta^2 \ln c_2 + \cdots$$

subject to

$$\begin{aligned} & & x_0 \text{ is given} \\ & (\forall t = 0, 1, 2, \ldots) & c_t + x_{t+1} = \gamma x_t \\ & (\forall t = 0, 1, 2, \ldots) & c_t \geq 0 \\ & (\forall t = 0, 1, 2, \ldots) & x_{t+1} \geq 0 \end{aligned}$$

Here, c_t $(t = 0, 1, 2, \ldots)$ denotes a consumption in the period t and x_t $(t = 0, 1, 2, \ldots)$ denotes the capital stock at the beginning of the period $t+1$ (that is, an investment made in the period t). An initial capital stock x_0 is exogenously given and positive. Also, β and γ are constants such that $\beta \in (0, 1)$ and $\gamma > 0$. By the assumption of $\beta \in (0, 1)$, we may invoke all the dynamic programming techniques because $\beta\gamma^{1-\theta} = \beta\gamma^{1-1} = \beta < 1$. (Recall that the logarithmic felicity function is in the CRRA class with $\theta = 1$.)

Answer all of the following questions.

(1) Derive Bellman's equation for this maximization problem.

(2) Derive the Euler in/equalities for this maximization problem.

(3) Prove that the transversality condition is always satisfied for any feasible path of this economy.

(4) Find the optimal consumption path precisely for this economy.

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(1) This optimization problem is know as the *log-Ak Model*. The Bellman's equation for this problem is:

$$V(x) = \max \left\{ \ln \left(\gamma x - x' \right) + \beta V(x') \middle| \, 0 \le x' \le \gamma x \right\} \,, \tag{1}$$

where x' denotes the capital stock in the beginning of the next period (that is, the investment made in the current period).

At this stage, the problem can be solved by the so-called "guess-and-verify" method, although I did not ask you to do that. Guess that $V(x) = A \ln x + B$ for some constants, A and B, that will be figured out later. Then, Equation (1) would take the form of

$$A \ln x + B = \max \left\{ \ln \left(\gamma x - x' \right) + \beta A \ln x' + \beta B \middle| \, 0 \le x' \le \gamma x \right\} \,. \tag{2}$$

The first-order necessary condition for the optimization in Equation (2) is $1/(\gamma x - x') = \beta A / x'$, which shows that

$$x' = \frac{\beta A}{1 + \beta A} \gamma x \,. \tag{3}$$

Plug this back into Equation (2), we get

$$\begin{aligned} A \ln x + B &= \ln \left(\gamma x - \frac{\beta A}{1 + \beta A} \gamma x \right) + \beta A \ln \left(\frac{\beta A}{1 + \beta A} \gamma x \right) + \beta B \\ &= \cdots\cdots\cdots \\ &= (1 + \beta A) \ln x + C \,, \end{aligned}$$

where C is some constant which I don't figure out though it is easy because it's needless for our objective. By equating the coefficient of $\ln x$, we get $A = 1 + \beta A$. Thus, we conclude that $A = 1/(1 - \beta)$. (Note that $\beta < 1$ by assumption.) By Equation (3),

$$x' = \frac{\beta/(1 - \beta)}{1 + \beta/(1 - \beta)} \gamma x = \beta \gamma x \,,$$

which shows that the portion β of the wealth of that period should be invested. You can verify the guess was correct because you can find B although you need to figure out C for that purpose.

Importantly, the procedure in the previous paragraph is correct because we know that the solution to Bellman's equation is the *true* value function when $\beta < 1$. I skipped this in the class. This is a recent result by Ozaki and Streufert. If you are interested in this, ask me directly.

(2) By the Benveniste-Scheinkman theorem or just looking at the value function above, we know that the value function is differentiable. Also, note that the Inada condition is satisfied:

$u'(0) = \infty$ and $f(0) = 0$, thus, no corner solutions occur. By differentiating Equation (1), we get

$$V'(x) = \frac{d}{dx} \ln\left(\gamma x - x'\right) = \frac{\gamma}{\gamma x - x'} \tag{4}$$

(I used the envelope theorem). On the other hand, the first-order necessary condition for the optimization problem in Equation (1), we have $1/(\gamma x - x') = \beta V'(x')$, which is combined with (4) to obtain $1/(\gamma x - x') = \beta\gamma/(\gamma x' - x'')$. In terms of the consumption, this is equivalent to

$$(\forall t \geq 0) \quad \frac{1}{c_t} = \frac{\beta\gamma}{c_{t+1}} \tag{5}$$

or $c_{t+1}/c_t = \beta\gamma$. This is the Euler equation for this problem.

(3) To state the transversality condition, we need to know the shadow price of the capital stock, $\lambda_t := V'(x_t)$. By what I did in Part (1), $V'(x_t) = A/x_t = 1/((1-\beta)x_t)$. The transversality condition is given by $\lim_{t\to\infty} \beta^t \lambda_t x_t = 0$. This holds now because

$$\lim_{t\to\infty} \beta^t \lambda_t x_t = \lim_{t\to\infty} \frac{\beta^t x_t}{(1-\beta)x_t} = \lim_{t\to\infty} \frac{\beta^t}{1-\beta} = 0$$

(recall that $\beta < 1$).

(4) Again, by what I did in Part (1), we know that along the optimal path, $(\forall t \geq 0)\ x_{t+1} = \beta\gamma x_t$, in particular, $x_1 = \beta\gamma x_0$, where x_0 is given for the problem. Therefore, $c_0 = \gamma x_0 - x_1 = \gamma x_0 - \beta\gamma x_0 = (1-\beta)\gamma x_0$. The Euler equation (4) is always a necessary condition for the optimization, and hence, $c_t^* = (\beta\gamma)^t(1-\beta)\gamma x_0$ and $x_t^* = (\beta\gamma)^t x_0$ (here, I denote the optimal path by *).

Importantly, in this model, the economy grows without a bound when $\beta\gamma > 1$, and we know that such a sustained growth path is certainly optimal as far as $\beta < 1$.