LECTURE NOTE FOR
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1 Setting Up Infinite-Horizon Optimization Problem

We address the following questions, which are solved in the remaining sections.

(1) Let a and b be real numbers (that is, $a, b \in \mathbb{R}$) and consider the closed interval $[a, b]$. More precisely, $[a, b] := \{c \in \mathbb{R} \,|\, a \leq c \leq b\}$. Also, let $g : [a, b] \to \mathbb{R}$ be a concave and differentiable function (both the right-derivative at a and the left-derivative at b exist). Characterize the maximum of g by using its derivatives. (By the so-called Weierstrass theorem, the maximum (or the maxima) always exists.)

(2) Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a felicity function and assume that u is specified by

$$(\forall c) \quad u(c) = c^{1-\rho}$$

where $\rho \in [0, 1)$. Note that u does not satisfy Inada conditions when $\rho = 0$. Also, let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a production function that is nondecreasing, concave, differentiable on its domain and satisfying $f(0) = 0$.

Then, consider the problem of maximizing

$$U({}_0\boldsymbol{c}) = U(c_0, c_1, c_2, \ldots) := \lim_{t \to +\infty} u(c_0) + \ \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^t u(c_t)$$

over the set of all consumption streams in the feasible set

$$\left\{ {}_0\boldsymbol{c} \in \mathbb{R}_+^\infty \;\middle|\; \left(\exists {}_1\boldsymbol{x} \in \mathbb{R}_+^\infty\right) \;\; c_0 + x_1 = f(x_0) \;\text{ and }\; (\forall t \geq 1)\; c_t + x_{t+1} = f(x_t) \right\},$$

given $\beta \in (0, 1)$ and $x_0 > 0$. Note that the limit always exists although it may be positive infinity.

(2-1) Derive the Euler equations for this problem. Since u may not satisfy Inada conditions, the equations may be inequalities in fact. Use (1) in such a case.

(2-2) Specify f by $(\forall x)\ f(x) = \gamma x$, where $\gamma > 0$. Determine whether an optimal consumption stream exists. If so, determine which feasible consumption streams satisfy the Euler equations, and then, determine which of these streams satisfy the transversality condition. On the basis of this information, derive the set of optimal streams. Conclude by determining the true value function.

The functional form of your answer will depend on the parameters β, γ and ρ. For instance, the assumption $\rho > 0$ leads to a case which is very different from the case specified by $\rho = 0$. Accordingly, break the problem into cases defined by the assumptions you place on the parameters.

(2-3) Specify f by $(\forall x)\ f(x) = x^{\alpha}$, where $\alpha \in (0, 1)$. Repeat the exercises of Question (2-2). That is, determine whether an optimal consumption stream exists. If so, determine which feasible consumption streams satisfy the Euler equations, and then, determine which of these streams satisfy the transversality condition. On the basis of this information, derive the set of optimal streams. Conclude by determining the true value function. (Break the problem into cases because functional forms depend on parameters.)

2 The Kuhn-Tucker Theory

Assume that $a < b$. (Otherwise, the problem would be trivial.) Then, the problem will be restated formally as

$$\max g(x) \ \text{ subject to } x \geq a \ \text{ and } \ x \leq b\,.$$

Let $L(x; \lambda_1, \lambda_2)$ be the *Lagrangian* defined by

$$L(x; \lambda_1, \lambda_2) = g(x) + \lambda_1(x - a) + \lambda_2(b - x)\,.$$

Since g is assumed to be concave, the necessary *and* sufficient conditions for x^* to be optimal is as follows: $(\exists \lambda_1^*, \lambda_2^*)$

$$\left.\frac{\partial L}{\partial x}\right|_{(x;\lambda_1,\lambda_2)=(x^*;\lambda_1^*,\lambda_2^*)} = g'(x^*) + \lambda_1^* - \lambda_2^* = 0 \tag{1}$$

$$\lambda_1(x^* - a) = 0 \ \text{ and } \ \lambda_2(b - x^*) = 0 \tag{2}$$

$$\lambda_1^* \geq 0 \ \text{ and } \ \lambda_2^* \geq 0 \tag{3}$$

$$x^* - a \geq 0 \ \text{ and } \ b - x^* \geq 0\,. \tag{4}$$

If $(x^*; \lambda_1^*, \lambda_2^*)$ is an optimum, it satisfies (1) - (4). Furthermore, if it satisfies (1) - (4), it *must* be an optimum. (To be precise, for (1) - (4) to become necessary conditions for the optimality, we need the so-called *constraint qualification*. There are many versions of the constraint qualification and I totally ignore this thing in this answer key.)

Conditions (2) are known as the *complementary slackness*. Conditions (3) require that the *Lagrange multipliers* should be non-negative. Conditions (4) are simply the original constraints.

The lagrange multiplier λ_1 is the "shadow price" of a in the sense that if the constraint $x \geq a$ will be slack slightly, that is, when a decreases, the optimized value of g will be increased by λ_1. A similar interpretation holds for λ_2.

You may casually restate these conditions without using λ's like x^* is optimal if $g'(x^*) = 0$ and $a < x^* < b$ or $g'(x^*) \leq 0$ and $x^* = a$ or $g'(x^*) \geq 0$ and $x^* = b$. I am afraid, though, that you would get stuck in what follows if you did this.

(By the way, I have wrote quite a good lecture note about the Kuhn-Tucker theory, that is, something about the necessity and the sufficiency of the above conditions for the optimality, and I am looking for somebody who may typeset this note by TEX.)

3 The Euler Equations

In this section, assume that the value function for this problem exists and is finite-valued. Otherwise, the following argument would not make any sense. If the optimal consumption path may attain the utility of the positive infinity, we discuss that case separately.

The Bellman's equation for this problem is given by

$$V(x_t) = \max \left\{ \, u\left(f(x_t) - x_{t+1}\right) + \beta V(x_{t+1}) \,|\, 0 \leq x_{t+1} \leq f(x_t) \, \right\} \, ,$$

which is differentiable by the Benveniste-Scheinkman Theorem. In order to derive the first-order necessary condition for the maximization problem defining V, let L be the Lagrangian:

$$L(x_t, x_{t+1}) = u\left(f(x_t) - x_{t+1}\right) + \beta V(x_{t+1}) + \lambda_t^x x_{t+1} + \lambda_t^c (f(x_t) - x_{t+1}) \, .$$

Then, Condition (1) for this maximum problem is

$$\frac{\partial L}{\partial x_{t+1}} = -u'(c_t) + \beta V'(x_{t+1}) + \lambda_t^x - \lambda_t^c = 0 \, . \tag{5}$$

Here, λ_t^x is the shadow price corresponding to the constraint: $x_{t+1} \geq 0$, and λ_t^c is the shadow price corresponding to the constraint: $c_t = f(x_t) - x_{t+1} \geq 0$. Importantly, I admit that the maximum may take place at the "corners." Equation (5) holds with an *equality*, and if the interiority of the maximum is guaranteed, say, by the Inada conditions, Equation (5) then turns out to be

$$-u'(c_t) + \beta V'(x_{t+1}) = 0 \, . \tag{6}$$

Otherwise, the equality in (6) would never be guaranteed. (We know that λ_t^x, $\lambda_t^c \geq 0$ by (3). But, this can't determine the direction of the inequality that would arise when the "corner" solutions were the case.)

By the envelop theorem, it holds that

$$V'(x_t) = \frac{\partial L(x_t, x_{t+1})}{\partial x_t} = u'(c_t) f'(x_t) + \lambda_t^c f'(x_t) \, . \tag{7}$$

Note that the derivative of the value function is the *current*, not present, shadow price of the capital stock x_t, that is, the marginal value of x_t in the unit of utility that is measured at time t. We denote it by λ_t, and thus, $V'(x_t) = \lambda_t$. By proceeding time for one period in Equation (7), we have reached

$$V'(x_{t+1}) = u'(c_{t+1})f'(x_{t+1}) + \lambda^c_{t+1}f'(x_{t+1}) = \left(u'(c_{t+1}) + \lambda^c_{t+1}\right) f'(x_{t+1}) = \lambda_{t+1}\,, \tag{8}$$

from which, together with Equation (5), we get

$$-u'(c_t) + \beta\left(u'(c_{t+1}) + \lambda^c_{t+1}\right) f'(x_{t+1}) + \lambda^x_t - \lambda^c_t = 0\,,$$

which may be rearranged to finally obtain THE *Euler equation*:

$$u'(c_t) + \lambda^c_t = \beta\left(u'(c_{t+1}) + \lambda^c_{t+1}\right) f'(x_{t+1}) + \lambda^x_t\,. \tag{9}$$

Because the shadow prices may not be zero in general, the Euler equality does not hold with an equality when we dispense with shadow prices. Thus, the Euler equation is the Euler *inequality* in general (if we don't use the shadow prices). While we know that the shadow prices are non-negative by (3), this fact does not determines the direction of the inequality in the Euler inequality.

4 General Remarks

Under the differentiability assumptions, the optimality and the finiteness of the utility along the optimal path imply that the Euler equation *must* hold. On the other hand, the concavity, the Euler equations, the transversality condition and the *lower-convergence* imply that the path is certainly an optimum. (This is exactly what I proved in the class.) Note that the lower-convergence is automatically satisfied by all the models in this problem set.

Furthermore, if the *upper-convergence* is satisfied, an optimum *must* exist. (This is exactly what the handout I distributed in the class shows, which is based on Ozaki and Streufert, 1996.) Note that the upper-convergence is equivalent to the finiteness of the utility when the utility function is time-separable as in this problem set. (See Streufert, 1990.)

5 The Model with Linear Production Function: $F(x) = \gamma x$

In this case, an economy may grow without a bound. We divide this case into the two sub-cases.

5.1 The Case Where $\rho = 0$

In this case, the Inada condition is not met, and hence, a solution may take place at the "corners." We further divide this case into the three sub-cases.

5.1.1 $\beta\gamma > 1$

See the case of $\beta\gamma^{1-\rho} > 1$ below. (Set $\rho = 0$ there.) The value function is given by $J^*(x) = +\infty$.

5.1.2 $\beta\gamma = 1$

In this case, *any* feasible consumption path attains the identical utility number given by γx_0, and hence, the value function is given by $J^*(x) = \gamma x$. I could show that any feasible path satisfies both the Euler equations and the transversality condition. Therefore, any feasible path is in fact an optimum. It seems boring to me doing all this and I skip it.

5.1.3 $\beta\gamma < 1$

In this case, the optimal consumption path is $(\gamma x_0, 0, 0, \ldots)$, obtaining the utility number of γx_0. This is the *unique* optimum since you would get worse if you chose any other feasible consumption path. Thus, the value function is $J^*(x) = \gamma x$. I could show that the path, $(\gamma x_0, 0, 0, \ldots)$, satisfies both the Euler equations and the transversality condition. Therefore, the path is in fact the optimum. It seems boring to me doing all this and I skip it again.

5.2 The Case Where $\rho \in (0, 1)$

We further divide this case into the three sub-cases.

5.2.1 $\beta\gamma^{1-\rho} > 1$

Let $\theta \in (0, 1)$. Then, consider the path define by

$$(\forall t \geq 0) \quad c_t = (1-\theta)\gamma x_t \quad \text{and} \quad x_{t+1} = \theta\gamma x_t \,.$$

Note that this is a path that is feasible from x_0 under the current technology. Then, the utility is now given by

$$\begin{aligned} U(\boldsymbol{c}) &= ((1-\theta)\gamma x_0)^{1-\rho} + \beta\left((1-\theta)\gamma\theta\gamma x_0\right)^{1-\rho} + \beta^2\left((1-\theta)\gamma\left(\theta\gamma\right)^2 x_0\right)^{1-\rho} + \cdots \\ &= ((1-\theta)\gamma x_0)^{1-\rho}\left(1 + \beta\left(\theta\gamma\right)^{1-\rho} + \beta^2\left(\left(\theta\gamma\right)^2\right)^{1-\rho} + \cdots\right) \\ &= ((1-\theta)\gamma x_0)^{1-\rho}\left(1 + \beta\left(\theta\gamma\right)^{1-\rho} + \left(\beta\left(\theta\gamma\right)^{1-\rho}\right)^2 + \cdots\right) . \end{aligned}$$

Therefore, the utility is $+\infty$ if and only if

$$\beta\left(\theta\gamma\right)^{1-\rho} \geq 1 \,,$$

which in turn is equivalent to

$$\theta \geq \frac{1}{\beta^{1/(1-\rho)}\gamma} = \left(\frac{1}{\beta\gamma^{1-\rho}}\right)^{1/(1-\rho)}$$

where the last term is less than 1 by the assumption of $\beta\gamma^{1-\rho} > 1$. We conclude that we can attain $+\infty$ as the utility number by choosing θ so that

$$\theta \in \left[\frac{1}{\beta^{1/(1-\rho)}\gamma}, 1\right) .$$

Thus, you can find uncountably many feasible consumption paths that attain the positive infinity as a utility number. Therefore, there exist uncountably many optima and the value function is given by $J^*(x) = +\infty$.

5.2.2 $\beta\gamma^{1-\rho} = 1$

Because $\beta < 1$, this case occurs only when $\gamma > 1$. Then, $\gamma^{1-\rho} < \gamma$ since $\rho \in (0, 1)$. Thus, we conclude that $\beta\gamma > 1$ in this case.

Define the consumption path $\boldsymbol{c}^t$ by $\boldsymbol{c}^t = (0, \ldots, 0, \gamma^{t+1}x_0, 0, \ldots,)$ where $c_t = \gamma^{t+1}x_0$. Note that $\boldsymbol{c}^t$ is certainly feasible from x_0. Then, $U(\boldsymbol{c}^t) = \beta^t c_t = (\beta\gamma)^t \gamma x_0$. Therefore, we see that $\lim_{t\to\infty} U(\boldsymbol{c}^t) = +\infty$ since $\beta\gamma > 1$.

I do <u>not</u> know whether or not there exists a feasible consumption path that attains the positive infinity exactly. Depending on this, the value function either is given by $J^*(x) = +\infty$ (when such a path exists) or does not exist (when such a path does not exist). (The Euler equation is irrelevant now since the finiteness is now violated.)

5.2.3 $\beta\gamma^{1-\rho} < 1$

First note that any feasible consumption path generates a (finite) utility that is uniformly less than some constant. (You can spell out this constant easily by calculating $U(\gamma x_0, \gamma^2 x_0, \ldots)$.) Also, note that the Inada condition is now met. Therefore, the Euler equation turns out to be

$$u'(c_t) = \beta u'(c_{t+1}) f'(x_{t+1}) . \tag{10}$$

Under the current specifications of u and f, this is identical to

$$c_t^{-\rho} = \beta\gamma c_{t+1}^{-\rho}$$

that leads to

$$\frac{c_{t+1}}{c_t} = (\beta\gamma)^{1/\rho} . \tag{11}$$

Any optimal consumption path *must* solve this first-order difference equation.

From the feasibility conditions, $x_1 = \gamma x_0 - c_0$. Then,

$$x_2 = \gamma x_1 - c_1 = \gamma^2 x_0 - \gamma c_0 - c_1 = \gamma^2 x_0 - \gamma c_0 - (\beta\gamma)^{1/\rho} c_0$$

where the last equality is derived by the Euler equation. Furthermore,

$$x_3 = \gamma x_2 - c_2 = \gamma^3 x_0 - \gamma^2 c_0 - \gamma (\beta\gamma)^{1/\rho} c_0 - c_2 = \gamma^3 x_0 - \gamma^2 c_0 - \gamma (\beta\gamma)^{1/\rho} c_0 - \left((\beta\gamma)^{1/\rho}\right)^2 c_0 \, .$$

By induction, we have

$$\begin{aligned} x_t &= \gamma^t x_0 - \left(\sum_{i=1}^{t} \gamma^{t-i} \left((\beta\gamma)^{1/\rho}\right)^{i-1}\right) c_0 \\ &= \gamma^t x_0 - \gamma^{t-1} \left(\sum_{i=1}^{t} \left((\beta\gamma^{1-\rho})^{1/\rho}\right)^{i-1}\right) c_0 \\ &= \gamma^{t-1} \left(\gamma x_0 - \left(\frac{1 - \left((\beta\gamma^{1-\rho})^{1/\rho}\right)^t}{1 - (\beta\gamma^{1-\rho})^{1/\rho}}\right) c_0\right) \end{aligned}$$

Thus, the transversality condition turns out to be

$$\begin{aligned} 0 &= \lim_{t\to\infty} \beta^t \lambda_t x_t \\ &= \lim_{t\to\infty} \beta^t u'(c_t) f'(x_t) x_t \\ &= \lim_{t\to\infty} \beta^t (1-\rho) c_t^{-\rho} \gamma x_t \\ &= \lim_{t\to\infty} \beta^t (1-\rho) \left(\left((\beta\gamma)^{1/\rho}\right)^t c_0\right)^{-\rho} \gamma x_t \\ &= \lim_{t\to\infty} \beta^t (1-\rho) \left(\left((\beta\gamma)^{1/\rho}\right)^t c_0\right)^{-\rho} \gamma \cdot \gamma^{t-1} \left(\gamma x_0 - \left(\frac{1 - \left((\beta\gamma^{1-\rho})^{1/\rho}\right)^t}{1 - (\beta\gamma^{1-\rho})^{1/\rho}}\right) c_0\right) \\ &= \lim_{t\to\infty} (1-\rho) c_0^{-\rho} \left(\gamma x_0 - \left(\frac{1 - \left((\beta\gamma^{1-\rho})^{1/\rho}\right)^t}{1 - (\beta\gamma^{1-\rho})^{1/\rho}}\right) c_0\right) \\ &= (1-\rho) c_0^{-\rho} \left(\gamma x_0 - \left(1 - (\beta\gamma^{1-\rho})^{1/\rho}\right)^{-1} c_0\right) . \end{aligned}$$

Therefore, the transversality condition implies that

$$c_0^* = \left(1 - (\beta\gamma^{1-\rho})^{1/\rho}\right) \gamma x_0 \, .$$

Since the Euler equation and the trasversality condition characterize the optimal path, the consumption path defined by $\boldsymbol{c}^* = (c_0^*, c_1^*, c_2^*, \ldots)$, where, for $t \geq 1$, c_t^* is defined by c_0^* and the Euler equation, *is* certainly the optimum.

I did not show the necessity of the transversality condition for the optimality in my lecture, and I don't know if there might exist another optimal consumption path or not.

The value function is given by

$$J^*(x) = \sum_{t=0}^{\infty} \beta^t (c_t^*)^{1-\rho} = \sum_{t=0}^{\infty} \beta^t \left((\beta\gamma)^{1/\rho} \right)^t c_0^* = \left(1 - \left(\beta\gamma^{1-\rho} \right)^{1/\rho} \right)^{-\rho} \gamma^{1-\rho} x_0^{1-\rho} .$$

Note that the optimal consumption path is steadily growing whenever

$$\frac{1}{\beta} < \gamma < \left(\frac{1}{\beta} \right)^{1/(1-\rho)} ,$$

where the second inequality holds by the assumption of this subsection. The whole inequalities define the nonempty open interval since $1/\beta > 1$ and $\rho \in (0, 1)$.

6 The Model with Concave Production Function: $F(x) = x^\alpha$

Note that the production function intersects the 45-degree line and thus the utility function over the feasible set is bounded given x_0 since the consumer discounts the future ($\beta \in (0, 1)$).

6.1 The Case Where $\rho = 0$

The Inada condition is *not* satisfied. Thus, the Euler equation is given by

$$1 + \lambda_t^c = \alpha\beta \left(1 + \lambda_{t+1}^c \right) x_{t+1}^{\alpha-1} + \lambda_t^x \tag{12}$$

(specify u and f in (9)). (You can not dispense with the shadow prices since some of the constraints may be binding.)

Let $(\bar{c}, \bar{x})$ be the steady state *that satisfies the Euler equations*. That is, $c_t = c_{t+1} = \bar{c} > 0$, $x_t = x_{t+1} = \bar{x} > 0$, $\bar{c} + \bar{x} = \bar{x}^\alpha$, and $1 = \alpha\beta\bar{x}^{\alpha-1}$. (At the steady state, the shadow prices are zero since any constraint is not binding there.) From these, we get $\bar{x} = (\alpha\beta)^{1/(1-\alpha)} < 1$ and $\bar{c} = (1 - \alpha\beta)(\alpha\beta)^{\alpha/(1-\alpha)}$.

Assume that $x_0 < \bar{x}^{1/\alpha}$. Let $\hat{x}$ be the *pure-accumulation* path. That is, $\hat{\boldsymbol{x}} = (x_0, \hat{x}_1, \hat{x}_2, \ldots) = (x_0, x_0^\alpha, (x_0^\alpha)^\alpha, \ldots)$. Since $\lim \cdots ((x_0)^\alpha)^\alpha \cdots = 1 > \bar{x}$, there exists the finite *stopping time* T such that

$$T = \min \left\{ t \in \mathbb{N} \,|\, \hat{x}_t^\alpha \geq \bar{x} \right\} .$$

Then, define the capital path $\boldsymbol{x}^*$ by

$$\boldsymbol{x}^* = (x_0, \hat{x}_1, \ldots, \hat{x}_{T-1}, \hat{x}_T, \bar{x}, \bar{x}, \ldots) ,$$

Finally, define the consumption path $\boldsymbol{c}^*$ by

$$\boldsymbol{c}^* = (0, 0, \ldots, 0, \hat{x}_T^\alpha - \bar{x}, \bar{c}, \bar{c}, \ldots) ,$$

where $c_T^* = \hat{x}_T^\alpha - \bar{x}$. (Intuitively, you accumulate until you will have reached the steady state and then you stay there.)

Note that $(\boldsymbol{c}^*, \boldsymbol{x}^*)$ thus defined satisfies the Euler equation. To see this fact, first note that $(\forall t)\ x_t^* > 0$ and hence that $(\forall t)\ \lambda_t^x = 0$ by the complementary slackness, (2). Thus, (12) can be slightly more simplified to

$$1 + \lambda_t^c = \alpha\beta \left(1 + \lambda_{t+1}^c\right) x_{t+1}^{\alpha-1}\,. \tag{13}$$

Second note that $c_T^* > 0$ and thus $\lambda_T^c = 0$ by the complementary slackness, (2). (There is a possibility that $c_T^* = 0$, but I ignore this.) Therefore, it must be the case that

$$1 + \lambda_{T-1}^c = \alpha\beta \left(1 + \lambda_T^c\right) \left(x_T^*\right)^{\alpha-1} = \alpha\beta \left(x_T^*\right)^{\alpha-1}\,.$$

Then, define λ_{T-1}^c by

$$\lambda_{T-1}^c = \alpha\beta \left(x_T^*\right)^{\alpha-1} - 1 = \alpha\beta \left(\hat{x}_T\right)^{\alpha-1} - 1 > \alpha\beta \left(\bar{x}\right)^{\alpha-1} - 1 = 0\,.$$

Then, define λ_{T-2}^c by

$$\begin{aligned}\lambda_{T-2}^c &= \alpha\beta \left(1 + \lambda_{T-1}^c\right) \left(x_{T-1}^*\right)^{\alpha-1} - 1 = \alpha\beta \left(x_{T-1}^*\right)^{\alpha-1} - 1 = \alpha\beta \left(\hat{x}_{T-1}\right)^{\alpha-1} - 1 \\ &> \alpha\beta \left(\bar{x}\right)^{\alpha-1} - 1 = 0\,,\end{aligned}$$

and so on. Thus, $(\boldsymbol{c}^*, \boldsymbol{x}^*)$ satisfies the Euler equations.

Furthermore, it clearly satisfies the transversality condition since it is a bounded path. Hence, we conclude that $(\boldsymbol{c}^*, \boldsymbol{x}^*)$ is certainly the optimum. The value function is given by

$$J^*(x) = U(\boldsymbol{c}^*) = \beta^T \left(\hat{x}_T^\alpha - \bar{x}\right) + \beta^{T+1} \frac{\bar{c}}{1-\beta}\,.$$

I do not know if there are any other optimal path or not because I did not show that $(\boldsymbol{c}^*, \boldsymbol{x}^*)$ is the only path that satisfies the Euler equation. In any case, this is an example of economies where the "turnpike" property holds. (The *turnpike property* says that it is optimal to get to the optimal steady state as soon as possible and to stay there as long as possible, in this case, to stay there forever.)

The case where $x_0 > \bar{x}^{1/\alpha}$ can be handled in quite a similar way and I skip it.

6.2 The Case Where $\rho \in (0, 1)$

The Inada condition is now met, and thus, the optimal path is the interior one. The Euler equation is given by (10) and is reduced to

$$c_t^{-\rho} = \alpha\beta c_{t+1}^{-\rho} x_{t+1}^{\alpha-1}\,. \tag{14}$$

Also, from the feasibility condition, it must be the case that

$$c_t + x_{t+1} = \left(x_t\right)^\alpha\,.$$

Given x_0, these constitute a system of a couple of the first-order difference equations with one boundary condition. Unfortunately though, I do not know the closed-form solution of this system unless $x_0 = \bar{x}$ by accident.

What I can say about this problem is that the optimum certainly exists and it satisfies (14). The latter claim follows immediately since the Euler equation is a necessary condition for the optimality. The existence of the optimum follows from the upper-convergence (see the 2nd paragraph of Section 3 above) and the upper-convergence follows since the utility is finite (see the 2nd paragraph of Section 3 above again).

7 References

Ozaki, Hiroyuki and Peter A. Streufert (1996): "Dynamic programming for non-additive stochastic objectives," *Journal of Mathematical Economics* 25, 391-442.

Peter A. Streufert (1988): Lectures given by P. A. Streufert at University of Wisconsin, Madison. (This answer key owes this lecture heavily although any remaining error is solely mine.)

Peter A. Streufert (1990): "Stationary recursive utility and dynamic programming under the assumption of biconvergence, " *Review of Economic Studies* 57, 79-97.