Subjective and Objective Probabilities in Representation of Preferences under Uncertainty*

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Abstract

Dana Scott provides a necessary and sufficient condition for a given probability capacity can be decomposed into a unique additive probability and a unique strictly increasing distortion function. Given this result, we characterize a family of preferences defined over the lottery acts (the Anscombe-Aumann acts), which we call *rank-dependent subjective expected utility* (RDSEU): The Choquet expected utility (CEU) preferences where a probability capacity is decomposed into an additive probability and a distortion function, both of which are endogenously derived from the preference. Furthermore, we clarify the relationship between this family of preferences and the so-called probabilistically sophisticated (PS) preferences. Specifically, a decision maker's preference is represented by RDSEU if and only if it is represented by CEU on the Anscombe-Aumann acts and by PS when its domain is restricted to the Savage acts. Moreover, if, in addition, the RDSEU is representable by PS on the entire Anscombe-Aumann acts, it turns out that it must be a subjective expected utility (CEU) on the entire domain. We thus uncover how subjective and objective probabilities are different not only in their concepts but also in their implications for preference representation.

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1 Introduction

Since the pioneering work by David Schmeidler (1982, 1989) appeared,[1] the Choquet expected utility (CEU) theory has been drastically prevailing in almost all the realms of economics analyzing *uncertain* phenomena that cannot be reduced to mere risky situations. Here, risky situations refer to those that can be described by a *single* objective probability measure. Not only that the CEU resolves the famous Ellsberg's (1961) paradox, which had been casting long-lasting doubt on the legitimacy of Savage's (1954) subjective expected utility (SEU) theory as a theory toward uncertainty, but also that it has been applied to the more concrete economic models in the hope of explaining the various economic phenomena observed in the real world that cannot be explained by the SEU though, including portfolio inertia (Dow and Werlang, 1992), high volatility in asset prices (Epstein and Wang, 1994; 1995), the reversal movements between the reservation wage and the degree of uncertainty in the job search (Nishimura and Ozaki, 2004), among many others.

Despite a denial of a 'single' probability measure in the theory of uncertainty, the probability measure still plays two important roles there. The first role is a purely objective one, which is referred to as *objective* probability. This type of probability appears as a part of the object of choice.

To illustrate the objective probability, consider the following simple choice situation. Suppose that the agent is contemplating tomorrow's plan and she has finally decided to go to shopping if it will be sunny; otherwise, she will postpone it and then flip a coin. She will watch the movie on Netflix if the coin's landing on the head, and she will reluctantly do homework if the coin's landing on the tail. The contingent plan described here is an example of the so-called *act*. Further, the act there includes flipping a coin. Such an act was first introduced by Anscombe and Aumann (1963) in the context of the subjective expected utility (SEU), and it is called an *Anscombe-Aumann act* (AA-act). Importantly, in this AA-act, whether to watch the movie or do homework is determined by flipping a coin, and the agent can*not* control this probability, for which reason this probability is purely objective. More generally, the AA-act is an act whose execution should require a randomizing device like a coin, a dice, or a roulette wheel.

There are at least two motivations for introducing objective probability to the theory of uncertainty. The first is that it is quite plausible to consider the AA-acts in the real world. For

[1]Schmeidler (1982) was circulated as a discussion paper, and then it was published in 1989.

example, we can consider a contingent plan (that is, an act) in which the agent is urged to buy lottery tickets upon the occurrence of some state. The second is more essential. Proving the CEU representation theorem by introducing the AA-acts exactly as Schmeidler (1982, 1989) did is much simpler than doing this by using only the *Savage act*, which is a contingent plan where the randomizing device is missing, exactly as Gilboa (1987) did. This can be directly seen by comparing these two proofs and can be understood intuitively if we recognize the fact that the set of AA-acts is larger than that of Savage acts, and thus the axioms imposed on the primitive preference are much stronger in Schmeidler's AA-act framework.[2]

The second out of the two roles is a purely subjective one, which is called *subjective* probability, in order to emphasize the difference in nature between this and the objective probability we mentioned before. This probability reflects the agent's subjective view toward the uncertainty she is faced with. This type of probability measure appears as a component of a probability capacity in the Choquet expected utility (CEU) developed by Schmeidler (1982, 1989) we talked about at the very start of this paper.

In the CEU representation with the AA-acts by Schmeidler (1982, 1989), the utility index of the outcome resulting from any act is evaluated by the so-called capacity, rather than the measure, in the form of Choquet integral. Here, the *capacity* is a set-function which is *not* necessarily additive over the union of mutually disjoint pair of sets, unlike the probability measure. Furthermore, the capacity that is derived in the CEU representation may be decomposed into two components: A strictly increasing function from $[0, 1]$ onto itself and a probability measure. Obviously, the composition of the former function, called *distortion function*, and the latter additive set-function constitutes a set-function which is not necessarily additive (*i.e.*, a capacity) unless the distortion function happens to be the identity function. The probability measure we are interested in for the current study is the second component of this composition.

We call the preferences that can be represented by the CEU with some decomposable capacity as above the *rank-dependent subjective expected utility* (RDSEU).[3] The main objectives of this

[2]This relationship exactly corresponds to the SEU with the AA-acts (Anscombe and Aumann, 1963) and that with the Savage acts (Savage, 1954). Strictly speaking, Anscombe and Aumann's (1963) original proof assumes a finite state space, and thus, its direct comparison with Savage's proof is impossible. For the detail and its direct comparison, see Theorem 3.8 in Nishimura and Ozaki (2017).

[3]The rank-dependent expected utility theory was originally introduced in the context of choice *under risk* initiated by von Neumann and Morgenstern (1953) rather than the theory of choice *under uncertainty* initiated by Savage (1954). Quiggin (1982) represents the preference of risk (charge) as the Choquet integral of the utility index with respect to the distorted risk, where the distortion is made by some non-linear function. Independently,

paper are to develop the RDSEU with the domain given by the AA acts and, building on the representation, to discuss the importance of distinguishing these two types of probabilities in representing preference under uncertainty.

As we discussed above, the probability incorporated into the AA acts is purely objective, meaning that it is given in the model, while the probability derived in the representation found by decomposing the capacity derived in the CEU representation is purely subjective. Therefore, a distortion of subjective probability in the RDSEU is meaningful as a part of an expression of the agent's subjective view toward uncertainty. However, if we would distort the objective probability in AA acts, it seems to be difficult to justify this procedure because the objective probability is given to the agent in a manner in which she cannot intentionally nor technically manipulate it. We thus should treat the subjective and objective probability differently in terms of distortion.

As our first main result, we offer some set of axioms that characterizes the preference as the one represented by the RDSEU over the set of *AA acts*.[4] To characterize the class of preferences, we follow the suggestion made by Machina and Schmeidler (1992, pp.768-9).[5] First, we impose the set of axioms for the CEU theorem à la Schmeidler (1982, 1989) on the primitive preference over the AA acts in order to derive the capacity with which the preference is represented by the Choquet integral. Second, we invoke Scott's (1964) theorem, which provides a necessary and sufficient condition, so-called *weak additivity*, for the capacity to be expressed as a composite of a probability measure and a strictly increasing distortion function.[6] Third and finally, we incorporate the weak additivity into the axiomatic system for the primitive preference over the AA acts. To the best of our knowledge, the characterization of the RDSEU with the preference

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Yaari (1987) considers a utility of money and shows that the preference can be represented equivalently by either some non-linear vNM index with the original distribution or the linear vNM index with some distorted distribution which is generated by applying some non-linear distortion function, resulting in the Riemann-*Stieltjes* integral, rather than the Riemann integral. His theory is thus often referred to as the *dual theory of risk*. Notice that, in the theory of choice under risk, the probability measures are given in the model as the objects of choice for the agent, and it does not matter whether it is subjective or it is objective. Hence, whether it is appropriate to distort the given probability or not might depend upon the nature of the risk involved in the given model.

[4]The axiomatization of RDSEU representation for the preferences with the domain of Savage acts was done by Nakamura (1995a), For his axioms, see Nakamura (1995a, b).

[5]Their suggestion is made in the context of Savage's (1954) framework with the Savage acts.

[6]Regardless of the great significance, Scott's theorem appeared in his unpublished manuscript (Scott, 1964). Gilboa (1985) "discovered" Scott's theorem and strengthened it by generalizing the strict increase of the distortion function to its weak increase.

defined on the set of AA acts, and in particular, the representation of some preference by means of Scott's theorem, seems to be new.

Our characterization of the RDSEU with the AA acts will add another important result to the theory of preference under uncertainty in which the representation of the preference involves both the subjective and objective probability, which includes the CEU representation with the AA acts by Schmeidler (1982, 1989) as well as the probability sophistication (PS) with the AA acts by Machina and Schmeidler (1995). Here, the PS is the preference that is representable by means of a single 'subjective' probability measure but which is not necessarily representable by the expected utility.

PS on Savage acts requires that the comparison of two acts must be consistent with the comparison of lotteries over outcomes generated by subjective probability. However, PS on AA acts additionally requires that the comparison must be also consistent with the objective probability governed by the acts. Based on this observation, as a second main result, we show that a preference relation is represented by RDSEU if and only if it is represented by CEU on the AA acts and represented by PS only on the Savage acts.

Moreover, if, in addition, the RDSEU is representable by PS on the entire Anscombe-Aumann acts, it turns out that it must be a subjective expected utility (SEU) on the entire domain. This result is in sharp contrast with the RDSEU representation by Nakamura (1995a) employing the Savage acts because only assuming the CEU and PS representability over the Savage acts does not imply the SEU representation for the Savage acts, while we are assuming a much stronger assumption of the CEU and PS representability for the *AA acts*, implying the SEU representation for the entire AA acts.

The rest of the paper is organized as follows. In Section 2, we provide some mathematical apparatus that is used in this paper. In Section 3, after defining the class of rank-dependent subjective expected utility (RDSEU), we give an axiomatic foundation of RDSEU. In Section 4, we consider the relationship between RDSEU and probabilistically sophisticated preferences. The proofs are relegated to the Appendix.

2 Mathematical and Decision-Theoretic Preliminaries

This section prepares some fundamental concepts which will be necessary to formally state and discuss the two main objectives of this paper described in the Introduction.

2.1 Basic Definitions of Probability Capacity and Choquet Integral

For more details on the concepts introduced in this subsection, see Nishimura and Ozaki (2017).

Let $(S, \mathscr{A})$ be a measurable space, where S is any infinite set and $\mathscr{A}$ is an algebra on S. Given such a measurable space, $(S, \mathscr{A})$, a set function, $\theta : \mathscr{A} \to \mathbb{R}$, is a *non-additive measure* or a *capactiy* if it satisfies both $\theta(\phi) = 0$ and $(\forall A, B \in \mathscr{A})\ A \subseteq B \Rightarrow \theta(A) \leq \theta(B)$. A capacity, θ, such that $\theta(S) < \infty$ is called a *finite* capacity or a *game*. A finite capacity that also satisfies $\theta(S) = 1$ is called a *probability* capacity. A probability capacity, θ, is *convex-ranged* by definition if $(\forall A \in \mathscr{A})(\forall r \in [0, \theta(A)])(\exists B \in \mathscr{A})\ B \subseteq A$ and $\theta(B) = r$.

A probability capacity, θ, on $(S, \mathscr{A})$ is a probability *charge* if in addition it satisfies the *finite-additivity*: $(\forall A, B \in \mathscr{A})\ A \cap B = \phi \Rightarrow \theta(A \cup B) = \theta(A) + \theta(B)$. Throughout this paper, a probability charge is often denoted by p or μ, instead of θ. Similarly to a probability capacity, a probability charge, μ, on $(S, \mathscr{A})$ is *convex-ranged* if $(\forall A \in \mathscr{A})(\forall r \in [0, \mu(A)])(\exists B \in \mathscr{A})\ B \subseteq A$ and $\mu(B) = r$.[7]

Let θ be a probability capacity on a measurable space, $(S, \mathscr{A})$. Then, for any real-valued $\mathscr{A}$-measurable bounded function, a, on S, the *Choquet integral with respect to* θ is denoted and defined by

$$\int_S a(s)\, d\theta(s) := \int_{-\infty}^{0} \Big(\theta(\{s|\, a(s) \geq z\}) - 1\Big) dz + \int_0^{\infty} \theta(\{s|\, a(s) \geq z\})\, dz\,, \tag{1}$$

where the integrals in the right-hand side are Riemann integrals in a wide sense, which can be easily verified to be well-defined.[8]

When a is a non-negative simple function, the definition (1) of the Choquet integral can be further simplified. To see this, let E^* be the indicator function of $E \in \mathscr{A}$, let $k \in \mathbb{N}$, let $+\infty > a_1 > a_2 > \cdots > a_k \geq 0$ and let $(\forall i \in \{1, 2, \ldots, k\})\ E_i := a^{-1}(\{a_i\})$. Here, the set, $\{a_1, \ldots, a_k\}$, is understood to be the range of f. Then, we can canonically write a as $(\forall s \in S)\ a(s) = \sum_{i=1}^k a_i E_i^*(s)$ and (1) is reduced into a more simple expression:

$$\begin{aligned} \int_S a(s)\, d\theta(s) &= \sum_{i=1}^{k} (a_i - a_{i+1})\theta\Big(\cup_{j=1}^{i} E_j\Big) \\ &= a_1\theta(E_1) + \sum_{i=2}^{k} a_i\bigg(\theta\Big(\cup_{j=1}^{i} E_j\Big) - \theta\Big(\cup_{j=1}^{i-1} E_j\Big)\bigg) \end{aligned} \tag{2}$$

where $a_{k+1} := 0$.

[7]The convex-rangedness of a probability charge is also referred to as the *strong non-atomicity*.

[8]In particular, "$-\infty + \infty$" never happens because of the boundedness of a.

2.2 A Brief Review of the CEU with AA-Acts

In this subsection, we briefly review the CEU preference with its domain given by the AA-acts, which would be of some help for our purposes.

Let X be any set, which we interpret as a set of (deterministic) *outcomes* or *prizes*. Also, let Y be a set of probability charges on $(X, 2^X)$ whose support is *finite*, each element of which is often referred to as a *simple lottery* on X. A mapping f from $(S, \mathscr{A})$ into Y is by definition a *simple Anscombe-Aumann act* or a *simple AA-act*, if its range is a finite subset of Y *and* it is $\mathscr{A}$-measurable when Y is endowed with the algebra generated by the discrete topology. In what follows, when we say AA-act, it means a simple AA-act unless otherwise stated. We denote the set of all AA-acts by L_0. Our main concern in this paper is the AA-act because it is defined by means of objective probabilities. However, we sometimes refer to also the *Savage act*: a mapping f from $(S, \mathscr{A})$ into X (*not* simple lotteries on it) such that its range is a finite subset of X *and* it is $\mathscr{A}$-measurable when X is endowed with the algebra generated by the discrete topology.[9] The set of all Savage acts is denoted by F_0. Because any element of X can be identified with the degenerate probability charge (point mass) concentrated at that element, we have that $F_0 \subseteq L_0$.

The primitive of the model is a binary relation, $\succsim$, on L_0. By the standard convention, we can induce the binary relation on Y from $\succsim$ and denote it by the same symbol, $\succsim$. A pair of two acts, $f, g \in L_0$, is by definition *co-monotonic* if $(\forall s, t \in S)\ f(s) \succ f(t) \Rightarrow g(t) \not\succ g(s)$, where $\succ$ denotes an asymmetric part of $\succsim$ and $\not\succ$ denotes the negation of $\succ$. The co-monotonicity is invoked in one of the axioms imposed in Schmeidler's theorem, which we describe in the next paragraph.

Schmeidler (1982, 1989) characterizes the *Choquet expected utility* (CEU): a binary relation[10], $\succsim$, on L_0 complies with some set of axioms if and only if it can be represented by

$$f \succsim g \iff \int_S u(f(s))\, d\theta(s) \geq \int_S u(g(s))\, d\theta(s),$$

where θ is a unique probability capacity on $(S, \mathscr{A})$, u is a non-constant real-valued affine function defined on Y which is unique up to a positive affine transformation, and the integrals are the Choquet integrals. In his theorem, both θ and u are endogenously derived from the primitive binary relation, $\succsim$, via the set of axioms he imposes on it.

[9]Precisely, this is a *simple* Savage act. Similarly to the AA-act, we suppress the adjective 'simple' unless otherwise stated.

[10]In fact, by one of his axioms, the binary relation always turns out to be a preference.

Note that the capacity derived in the above theorem reflects the decision-maker's view toward uncertainty in a purely *subjective* manner.

2.3 A Brief Review of the PS with AA-Acts

Another important characterization of a preference[11] under uncertainty where the domain of the preference is specified by the AA-acts is achieved by Machina and Schmeidler (1995). Here, we briefly review their result for later use.

According to Machina and Schmeidler (1995), we assume an algebra, $\mathscr{A}$, to be generated by a finite partition of S.[12] Given an AA-act, f, and a probability charge, μ, on $(S, \mathscr{A})$, define a simple lottery $p_{f,\mu} \in Y$ by

$$(\forall x \in X) \quad p_{f,\mu}(x) := \sum_{y \in f(S)} \mu\left(f^{-1}(\{y\})\right) \cdot y(x), \tag{3}$$

where $\cdot$ is the usual product, $y(x)$ abbreviates $y(\{x\})$, meaning the probability charge of $x \in X$ by a simple lottery y, and the summation is well-defined (*i.e.,* finite) because $\mathscr{A}$ is finite and f is simple. By the way of construction, $p_{f,\mu}$ is certainly a simple lottery on X. We say that $p_{f,\mu}$ is a simple lottery induced on X from both f and μ.[13]

Next, consider a mapping $V : Y \to \mathbb{R}$, called *preference functional*. The decision-maker is *probabilistically sophisticated* (PS) by definition if there exist a probability charge μ on $(S, \mathscr{A})$ and a preference functional $V : Y \to \mathbb{R}$, such that $(\forall f, g \in L_0)\ \ f \succsim g \iff V\left(p_{f,\mu}\right) \geq V\left(p_{g,\mu}\right)$.

Importantly, a preference functional V does not necessarily take the form of the expected utility functional with some utility index u. Furthermore, the probability charge, μ, for the PS decision-maker is purely *subjective* in nature, which reflects her view toward uncertainty, just like the probability capacity derived in Schmeidler's theorem. Because the AA-act is defined via *objective* probability charges, both subjective and objective probability charges show up in an essential way in the PS preference with AA-acts.

[11]Precisely, it should be written as a binary relation, but the fact similar to the footnote 10 also applies to the representation result below.

[12]Note that we invoke this assumption only when we mention the SP. For instance, when we characterize the RDSEU, this assumption is unnecessary.

[13]When an act, f, were a Savage act rather than an AA-act, the formula (3) would be somewhat simplified to

$$(\forall x \in X) \quad p_{f,\mu}(x) := \mu\left(f^{-1}(\{x\})\right). \tag{4}$$

Machina and Schmeidler (1995) develop a system of axioms that should be imposed on the decision-maker's preference with its domain specified by the AA-acts, characterizing her as PS with some unique probability charge, μ, and some utility functional, V. In addition, their axiomatic system forces V to be strictly monotonic in the sense defined below.

First, we define the concept of stochastic dominance. For any $p, q \in Y$, p *stochastically dominates* q if $(\forall x \in X)\ \sum_{\{i \,|\, x^i \precsim x\}} p^i \ \leq \sum_{\{j \,|\, z^j \precsim x\}} q^j$. Here, without loss of generality, we are writing as $p = (x^1, p^1; \ldots; x^m, p^m)$ and $q = (z^1, q^1; \ldots; z^n, q^n)$ with some integers, m and n. If a strict inequality holds there for some outcome x, p *strictly* stochastically dominates q.

Then, we use the stochastic dominance relation to define the strict monotonicity of V. Let f be any AA-act and let E be any set that constitutes the given finite partition of S (together with the other sets). We denote by y the single simple lottery which f takes on when $s \in E$. Also, we define another AA-act, g, by: $g(s) = y'$ if $s \in E$ and $g(s) = f(s)$ if $s \in S \backslash E$, where y' is any simple lottery. A preference functional, V, is *strictly monotonic* by definition if $V(p_{f,\mu}) \geq V(p_{g,\mu})$ whenever y stochastically dominates y' *and* if $V(p_{f,\mu}) > V(p_{g,\mu})$ whenever y *strictly* stochastically dominates y'.[14] Henceforth, we always assume that V is strictly monotonic when we mention the PS representation of the primitive preference

3 Characterizing the RDSEU with AA-Acts

This section attempts to achieve the first out of two objectives mentioned in the Introduction: the characterization of the rank-dependent subjective expected utility (RDSEU) with its domain given by the AA-acts.

To this end, the next subsection defines the decomposability of a probability capacity and then presents the necessary and sufficient conditions for the capacity to be decomposable, which plays a significant role in our objective of this section.

3.1 Decomposable Capacities and Scott's Theorem

Given a measurable space, $(S, \mathscr{A})$, we call a probability capacity, θ, on it as *strongly decomposable* (resp. *weakly decomposable*) if there exist a probability charge, p, on $(S, \mathscr{A})$ as well as a

[14] Even if the domain of the preference is confined to the Savage acts, the definition of the strict monotonicity of V remains as it was with (3) replaced by (4).

strictly (resp. *weakly*) increasing function such that $(\forall E \in \mathscr{A})\ \theta(E) = \gamma \circ p(E)$.[15] Regardless of whether it is either strictly or weakly increasing, the function, γ, which appears in the definition of the decomposability, will be referred to as a *distortion function*.

It is obvious that if a probability capacity is strongly decomposable, then it is also weakly decomposable, while the converse does not hold true in general. Furthermore, it is well-known that there exists some probability capacity which is *not* weakly decomposable and hence, nor strongly decomposable.[16]

A fundamental result of the probability capacity's decomposition is *Scott's theorem* (Scott, 1964; Gilboa, 1985), which characterizes the strong decomposability of the probability capacity. To state the theorem precisely, we introduce a condition named weak additivity: Given a measurable space, $(S, \mathscr{A})$, a probability capacity, θ, on it is *weakly additive* if, for any $A, B, E, F \in \mathscr{A}$ satisfying both $E \subseteq A \cap B$ and $F \subseteq S \setminus (A \cup B)$, it holds that $\theta(A) > \theta(B)$ whenever $\theta((A \setminus E) \cup F) > \theta((B \setminus E) \cup F)$ is true.

Basically, weak additivity requires that the set operation of subtracting a set from the intersection of the given two sets and adding another set to outside the union of the same two sets should not alter the "relative likelihood of occurrence" between the resultant two sets from that between the original two sets, where the likelihood is measured by θ. Then, Scott's theorem declares the following.

Theorem 1 (Scott)**.** *Let $\mathscr{A} := 2^S$. Then, for any convex-ranged probability capacity, θ, on $(S, \mathscr{A})$, it is weakly additive if and only if there exist a unique convex-ranged probability charge, p, on $(S, \mathscr{A})$ as well as a unique strictly increasing distortion function, $\gamma : [0, 1] \to [0, 1]$, such that $\theta = \gamma \circ p$.*

The theorem states that weak additivity is a necessary and sufficient condition for the given probability capacity to be *strongly* decomposable, where the decomposition is unique, with some additional assumptions. That is, $\mathscr{A}$ needs to be 2^S, and a probability capacity is assumed to be convex-ranged (and hence, S is forced to be an infinite set). For its proof, see Gilboa (1985).

[15]Note that it follows that both $\gamma(0) = 0$ and $\gamma(1) = 1$ by the definition of the probability capacity.

[16]See, for example, Chateauneuf (1991, p.364, Example 4).

3.2 A Simple Exposition of the RDSEU

Before presenting our main theorem of this section, we exhibit a preference that is represented by the RDSEU.

The RDSEU with AA-acts is a subclass of the CEU with AA-acts such that the probability capacity derived in the CEU representation is strongly decomposable, and the decomposition is unique in the sense of Scott's theorem (Theorem 1).

Formally, the preference is represented by the RDSEU by definition if and only if there exist a probability charge, p, on $(S, \mathscr{A})$, a strictly increasing distortion function, $\gamma : [0,1] \to [0,1]$, as well as a non-constant affine function, $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation such that for any pair of AA-acts, f and g, it holds that

$$f \succsim g \iff \int_S u(f(s))\,(\gamma \circ p)\,(ds) \geq \int_S u(g(s))\,(\gamma \circ p)\,(ds)\,, \tag{5}$$

where the integrals are the Choquet integrals.

As we already pointed out, the probability capacity in the CEU is subjective one. Therefore, the probability charge (together with the distortion function) that appears in the RDSEU is also a subjective one because it inherits subjective nature from the CEU representation. Therefore, both the *subjective* probability charge, p, in the representation and the *objective* probability charges embodied in each AA-act play substantial roles in the RDSEU, just like the PS preference with AA-acts.

3.3 The Representation Theorem

With all the main ingredients prepared, we are now ready to present the representation theorem of the RDSEU with AA-acts, which is new as far as the authors know.

Along the lines mentioned in the Introduction, we first characterize the preference by the CEU, and then we add the two new axioms in order to invoke Scott's theorem (Theorem 1) for the probability capacity derived in the CEU representation to be decomposed into the probability charge and the distortion function.

Therefore, the first five axioms in the following list are exactly the same as those of Schmeidler (1989). The roles played by the remaining two new axioms will be explained right after the exposition of the theorem.

Also note that in the list of the seven axioms, an AA-act, f, of the form:

$$f := \begin{bmatrix} x & \text{on} & A \\ y & \text{on} & S\backslash A \end{bmatrix}$$

denotes the 'binary' act such that $f(s) := x \in Y$ if $s \in A$ and $f(s) = y \in Y$ if $s \notin A$.

A1 (Ordering). $\succsim$ is complete and transitive.

A2 (Co-Monotonic Independence). For any mutually co-monotonic acts $f, g, h \in L_0$ and $\lambda \in (0, 1)$, $f \succ g \Rightarrow \lambda f + (1 - \lambda)h \succ \lambda g + (1 - \lambda)h$.

A3 (Continuity). For any $f, g, h \in L_0$, if $f \succ g$ and $g \succ h$, then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h$.

A4 (Monotonicity). For any $f, g \in L_0$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

A5 (Non-Degeneracy). There exist $f, g \in L_0$ such that $f \succ g$.

A6 (Smoothness). For any y, y' with $y \succ y'$, any $A \in 2^S$, and any $\alpha \in [0, 1]$, there exists $B \subseteq A$ such that $\alpha f^A_{y,y'} + (1 - \alpha)y' \sim f^B_{y,y'}$, where

$$f^A_{y,y'} := \begin{bmatrix} y & \text{on} & A \\ y' & \text{on} & S\backslash A \end{bmatrix}$$

and $f^B_{y,y'}$ are defined similarly with A replaced by B.

A7 (Rank-Dependent Comparative Probability). There exists a simple lottery $\underline{y} \in Y$ such that, for any $A, B, E, F \subseteq S$ with $E \subseteq A \cup B$, $F \subseteq S \setminus (A \cup B)$ and $x \in Y$ with $x \succ \underline{y}$,

$$\begin{bmatrix} x & \text{on} & (A\backslash E) \cup F \\ \underline{y} & \text{on} & S\backslash((A\backslash E) \cup F) \end{bmatrix} \succ \begin{bmatrix} x & \text{on} & (B\backslash E) \cup F \\ \underline{y} & \text{on} & S\backslash((B\backslash E) \cup F) \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x & \text{on} & A \\ \underline{y} & \text{on} & S\backslash A \end{bmatrix} \succ \begin{bmatrix} x & \text{on} & B \\ \underline{y} & \text{on} & S\backslash B \end{bmatrix}.$$

Our main theorem is stated as follows:

Theorem 2 (RDSEU). *Let $\mathscr{A} := 2^S$. A binary relation $\succsim$ on L_0 satisfies A1, A2, A3, A4, A5, A6 as well as A7 if and only if $\succsim$ is represented by the RDSEU in the sense of* (5) *with p convex-ranged.*

The proof is relegated to Appendix A.1. Here, we provide a sketch of the proof. We know from Schmeidler (1989) that the first five axioms characterize $\succsim$ as the CEU preference. However, the probability capacity derived from these axioms is *not* necessarily convex-ranged because of the AA-act framework he employs. This could be problematic for our purpose because an application of Scott's theorem requires the probability capacity to be convex-ranged in an essential way. Axiom A6 guarantees that the probability capacity derived from Axioms A1-A5 is convex-ranged even in the AA-act framework.

The basic idea embodied in A6 is that the decision-maker can "smoothly" make a convex combination between a binary act and a constant act better by making a better constant act in that binary act occurring on a bigger event. By this "smoothness," the resultant probability capacity can be shown to become convex-ranged.

Note that the convex-rangedness of the probability capacity inevitably requires that $\mathscr{A}$ be equal to 2^S, which is why we assume it at the beginning of the statement of Theorem 2. Note that because the CEU representation theorem with the AA-acts does not impose any restriction on $(S, \mathscr{A})$, the requirement of $\mathscr{A} = 2^S$ is safely assumed. In addition, S must be an infinite set for Axiom A6 to be meaningful.

Finally, Axiom A7 (as well as Axiom A6) seems to be new in the literature, and it enables us to invoke Scott's Theorem.

Consider a binary act consisting of a constant act $\underline{y}$, which may be called a reference point, and any constant act x which is strictly preferred to $\underline{y}$. Also, assume that

$$\begin{bmatrix} x & \text{on} & A \\ \underline{y} & \text{on} & S \backslash A \end{bmatrix} \succsim \begin{bmatrix} x & \text{on} & B \\ \underline{y} & \text{on} & S \backslash B \end{bmatrix}.$$

It is then natural to imagine that the decision-maker believes that A is equally or more likely to happen than B is. Now suppose that if $E \subseteq A \cap B$ is subtracted from both A and B and if $F \subseteq S \setminus (A \cup B)$ is added to both A and B, the decision-maker's belief is still the same as above. If such a belief structure of the decision-maker were translated into the language made of the concept of probability capacity, it would become a sufficient and necessary condition for the probability capacity to be decomposed into a strictly increasing distortion function and a probability charge by Scott's theorem. The requirement for the decision-maker's belief about the likelihood of events stated right above is nothing but Axiom A7.

4 The Relation among CEU, PS, and RDSEU

This section attempts to achieve the second out of two objectives mentioned in the Introduction: clarifying the role played by the objective probability charges embodied in each AA-act in the preferences under uncertainty, including the CEU, the PS as well as the RDSEU with all their domains specified by the AA-acts. In particular, we prove that within the class of the CEU preferences with the AA-acts, the PS and the subjective expected utility (SEU) are equivalent. To avoid being repetitious, the domain of each relevant preference is always the set of the AA-acts unless otherwise stated in the remainder of the paper.

The first result claims that the RDSEU is contained in the class of the PS.[17] This is somewhat intuitive because the RDSEU explicitly utilizes the probability charge, which is derived from the axioms.

Proposition 1. *If the binary relation $\succsim$ is the RDSEU, then it is also the PS when it is restricted on F_0.*

Next, we show that both the CEU and the PS "almost" imply the RDSEU. To put it another way, within the class of CEU, the PS and the RDSEU are "almost" equivalent. Here, we say "almost" because the PS is required only when the preference is restricted on the set of Savage acts, F_0, *and* we need some technical conditions.

Theorem 3. *Let $\mathscr{A} := 2^S$, suppose that a binary relation $\succsim$ is represented by the CEU with a convex-ranged probability capacity, and suppose that it is also represented by the PS with a convex-ranged probability charge when it is restricted on F_0,[18] then it is represented by the RDSEU.* (*Proof A.2 in Appendix*)

Combining this theorem with Proposition 1 immediately shows the next result.

Corollary 1 (Almost Equivalence between PS and RDSEU Given CEU)**.** *Assume that $\mathscr{A} = 2^S$ and that a binary relation $\succsim$ is represented by the CEU with a convex-ranged probability capacity. Then, $\succsim$ is represented by the PS with a convex-ranged probability charge when it is*

[17]For the related results, see also Grant and Kajii (2005) and Qu (2015).

[18]Note that in the framework employing only the Savage acts (that is, with no objective probability charges), the probability capacity derived in the CEU by Gilboa's (1987) axioms and the probability charge derived in the PS by Machina and Schmeidler's (1992) axioms are both automatically convex-ranged.

restricted on F_0 if and only if it is represented by the RDSEU with the convex-ranged probability charge. [19]

By virtue of the objective probability charges, the result can be much more sharpened up to the *exact* equivalence. Furthermore, the distortion function appearing in the RDSEU representation should be an identity mapping. More precisely, we can prove the next result.

Theorem 4 (Exact Equivalence between PS and SEU Given CEU)**.** *Suppose that a binary relation $\succsim$ is represented by the CEU. Then, $\succsim$ is the PS if and only if it is the SEU.* (*Proof A.2 in Appendix*)

Contrastingly, assuming that the preference is represented by the CEU only when it is restricted to the set of all Savage acts, F_0, is not strong enough to deduce the exact equivalence between the PS and the SEU. To see this fact, consider the next example of a preference which is represented by the CEU when it is restricted to F_0 and is also represented by the PS when such a restriction is removed, and all the AA-acts are taken into account. This preference is, however, not represented by the SEU, even if it is restricted to F_0.

Example 1. Let $(S, \mathscr{A})$ be any measurable space representing a state space, let μ be any simple probability charge on it, let X be any finite outcome space, let Y is the set of (simple) probability charges on X, and let $u : X \to \mathbb{R}$ be any vNM index. Without loss of generality, we can enumerate the elements of X so that $X = \{x_1, x_2, \ldots, x_n\}$ and $u(x_1) > u(x_2) > \cdots > u(x_n)$.

Given all these and any (simple) AA-act, $f : S \to Y$, the specific element of Y which is induced by μ and f is denoted and defined by

$$(\forall x \in X) \quad p_{f,\mu}(\{x\}) := \sum_{y \in f(S)} \mu\left(f^{-1}(\{y\})\right) \cdot y(\{x\})\,.$$

Finally, we define the preference functional $V : Y \to \mathbb{R}$ by

$$(\forall f \in L_0) \quad V(p_{f,\mu}) := u(x_1) - \sum_{k=1}^{n-1} \left(u\left(x_k\right) - u\left(x_{k+1}\right)\right) \gamma \left(\sum_{j=k+1}^{n} p_{f,\mu}\left(\left\{x_j\right\}\right)\right), \tag{6}$$

where $\gamma : [0,1] \to [0,1]$ is any *strictly increasing* surjective function.

The preference defined via V is certainly the desired example. For the detail, see Appendix A.3.

Summarizing the above discussions, we uncover how subjective and objective probabilities are different not only in their concept but also in their implication for preference representation.

[19] Note that Theorem 2 implies that the probability charge that appears in the RDSEU representation under the axioms there is convex-ranged.

APPENDIX

A Proofs

A.1 Proof of Theorem 2 (RDSEU)

Step 1 (Schmeidler's representation). First, we set $(S, \mathscr{A}) := (S, 2^S)$. Because Schmeidler's (1989) representation theorem imposes no restrictions on $\mathscr{A}$, this is totally permitted. Then, by his theorem, the binary relation $\succsim$ satisfies A1-A5 if and only if it is represented by

$$f \succsim g \;\Leftrightarrow\; \int_S u(f(s))\,\theta(ds) \geq \int_S u(g(s))\,\theta(ds)\,, \tag{7}$$

where θ is some unique probability capacity on $(S, 2^S)$, and $u : Y \to \mathbb{R}$ is some affine function that is unique up to a positive affine transformation.

Step 2 (Some useful intermediary facts in Schmeidler's representation theorem).[20] In this step, we list some useful results which appeared in the process of the proof of Schmeidler's theorem (1989). There exists an affine function $u : Y \to \mathbb{R}$ which represents the restriction of $\succsim$ on L_c. By the affinity and non-degeracy, we can always find $y^*, y_* \in Y$ such that $u(y^*) = 1$ and $u(y_*) = 0$. Furthermore, according to von Neumann-Morgenstern's theorem (1947), we can define a function $J : L_0 \to \mathbb{R}$ via u (see Nishimura and Ozaki, 2017, A.2.1, p.263). Define $K := u(Y)$ and let $B_0(K)$ be the set of 2^S-measurable K-valued simple functions on S. Also, define a function $U : L_0 \to B_0(K)$ by $(\forall f \in L_0)(\forall s \in S)\; U(f)(s) := u(f(s))$. Finally, we define a functional $I : B_0(K) \to \mathbb{R}$ by $(\forall a \in B_0(K))\; I(a) := J(U^{-1}(\{a\}))$, where I turns out to be well-defined. Therefore, by Theorem 2.4.5 (Schmeidler, 1986; Nishimura ana Ozaki, 2017, p.44), it holds that both $J(\cdot)$ and $I(U(\cdot))$ represent $\succsim$ on L_0 and $(\forall E)\; \theta(E) = I(E^*)$ where

$$(\forall a \in B_0(K)) \quad I(a) = \int_S a(s)\, d\theta(s)\,.$$

Step 3 (θ's convex-rangedness). This step proves that, given A1-A5, the derived probability capacity θ in (7) is convex-ranged if and only if A6 is satisfied. Note that, under A1 - A5, (7) shows that

$$\int_S u\left(\alpha f^A_{y,y'}(s) + (1-\alpha)y'\right)\theta(ds) = \alpha \int_S u\left(f^A_{y,y'}(s)\right)\theta(ds) + (1-\alpha)u(y')$$

$$= \; \alpha\left((u(y) - u(y'))\,\theta(A) + u(y')\right) + (1-\alpha)u(y') = \alpha\theta(A)u(y) + \left((1-\alpha\theta(A)\right)u(y')\,, \tag{8}$$

[20]For this step 2, see Nishimura and Ozaki (2017, A.2.5, p.268).

by the affinity of u, as well as it also shows that

$$\int_S u\left(f^B_{y,y'}(s)\right) = \theta(B)u(y) + (1-\theta(B))u(y') \,. \tag{9}$$

In deriving both (8) and (9), we invoked the definition of the Choquet integral.

In order to show the sufficiency of A6, let $y := y^*$ and $y' := y_*$, where $y^*, y_* \in Y$ are both derived in Step 2. Then, (8) and (9) are reduced to $\alpha\theta(A)$ and $\theta(B)$, respectively. Because A6 states that we could choose some subset B of A dependent on any $\alpha \in [0,1]$ so that $\theta(B) = \alpha\theta(A)$, θ must be convex-ranged.

In order to show the necessity of A6, take any $y, y' \in Y$ with $y \succ y'$, $A \in 2^S$, and $\alpha \in [0,1]$. If θ is convex-ranged, we can choose some B such that $B \subseteq A$ and $\theta(B) = \alpha\theta(A)$. Thus, we have (8) = (9) and we conclude that two acts evaluated by (8) and (9) are mutually indifferent, which completes the proof.

Step 4 (A binary act f^{E^*}). First, define an act f^{E^*} by

$$(\forall E \in 2^S) \quad f^{E^*} := \begin{bmatrix} y^* & \text{on} & E \\ y_* & \text{on} & S\backslash E \end{bmatrix},$$

where $y^*, y_* \in Y$ are the ones found in Step 2. The objective of this step is to show that

$$(\forall E \in 2^S) \quad J(f^{E^*}) = I(E^*) = \theta(E)\,.$$

By definition of U (Step 2), $U(f^{E^*})(s) = u(f^{E^*}(s)) = u(y^*) = 1$ if $s \in E$ and $U(f^{E^*})(s) = u(f^{E^*}(s)) = u(y_*) = 0$ if $s \in S\backslash E$ (we used Step 2 again). Therefore, $U(f^{E^*}) = E^*$, showing that together with the end of Step 2, $J(f^{E^*}) = I(U(f^{E^*})) = I(E^*) = \theta(E)$, where the second equality is proved right above in this step.

Step 5 (θ's weak additivity). This step proves that, given A1-A5, the derived probability capacity θ in (7) is weakly additive if and only if A7 is satisfied.

In order to the sufficiency, assume that A7 is satisfied and let $\underline{y} := y_*$, which is derived in Step 2. Further, take any $A, B, E, F \subseteq S$ and $x \in Y$ satisfying the conditions in A7. By the affinity of u, we can always set $x := y^*$ so that $u(x) = 1$. By the definition introduced in Step 4, note that

$$f^{((A\backslash E)\cup F)^*} = \begin{bmatrix} x & \text{on} & (A\backslash E)\cup F \\ \underline{y} & \text{on} & S\backslash((A\backslash E)\cup F) \end{bmatrix} \text{ and } f^{((B\backslash E)\cup F)^*} = \begin{bmatrix} x & \text{on} & (B\backslash E)\cup F \\ \underline{y} & \text{on} & S\backslash((B\backslash E)\cup F) \end{bmatrix}.$$

Then, Step 4 implies

$$\begin{aligned} f^{((A\backslash E)\cup F)^*} \succ f^{((B\backslash E)\cup F)^*} \quad &\Leftrightarrow \quad J\left(f^{((A\backslash E)\cup F)^*}\right) > J\left(f^{((B\backslash E)\cup F)^*}\right) \\ &\Leftrightarrow \quad \theta\left((A\backslash E)\cup F\right) > \theta\left((B\backslash E)\cup F\right)\,, \end{aligned} \tag{10}$$

where the first equivalence follows from the fact that J represents $\succsim$, which is mentioned in Step 2. Furthermore, by similar notations and reasoning, we have

$$\begin{bmatrix} x & \text{on} & A \\ \underline{y} & \text{on} & S\backslash A \end{bmatrix} \succ \begin{bmatrix} x & \text{on} & B \\ \underline{y} & \text{on} & S\backslash B \end{bmatrix} \Leftrightarrow f^{A^*} \succ f^{B^*} \Leftrightarrow \theta(A) > \theta(B)\,. \tag{11}$$

By (10), (11) as well as A7 together imply the weak additivity of θ. Because θ is convex-ranged by Step 4, Scott's Theorem (Theorem 1) proves that θ is strongly decomposable.

To shoe the necessity, suppose that θ is weakly additive and let $\underline{y} := y_* \in Y$. Further, take any $A, B, E, F \subseteq S$ and $x \in Y$ that satisfy the conditions in A7. Finally, define four binary acts f, g, f' as well as g' by

$$f := \begin{bmatrix} x & \text{on} & (A\backslash E) \cup F \\ \underline{y} & \text{on} & S\backslash((A\backslash E) \cup F) \end{bmatrix}, \quad g := \begin{bmatrix} x & \text{on} & (B\backslash E) \cup F \\ \underline{y} & \text{on} & S\backslash((A\backslash E) \cup F) \end{bmatrix},$$

$$f' := \begin{bmatrix} x & \text{on} & A \\ \underline{y} & \text{on} & S\backslash A \end{bmatrix} \quad \text{and} \quad g' := \begin{bmatrix} x & \text{on} & B \\ \underline{y} & \text{on} & S\backslash B \end{bmatrix}.$$

Now, assume that $f \succ g$. Then,

$$\begin{aligned} & f \succ g \Leftrightarrow J(f) > J(g) \Leftrightarrow \int_S u(f(s))d\theta > \int_S u(g(s))d\theta \\ \Leftrightarrow \quad & u(x) \int_{(A\backslash E)\cup F} d\theta > u(x) \int_{(B\backslash E)\cup F} d\theta \Leftrightarrow \theta\left((A\backslash E) \cup F\right) > \theta\left((B\backslash E) \cup F\right) \\ \Rightarrow \quad & \theta(A) > \theta(B) \Leftrightarrow \int_A d\theta > \int_B d\theta \Leftrightarrow \int_S u(f'(s))d\theta > \int_S u(g'(s))d\theta \\ \Leftrightarrow \quad & J(f') > J(g') \Leftrightarrow f' \succ g', \end{aligned}$$

where the first and last equivalences hold by the fact that J represents $\succ$ (Step 2); the second and seventh equivalences hold by the definition of J (see Nishimura and Ozaki, 2017, p.269); the third and sixth equivalences hold by the definition of the binary acts, f, g, f' and g'; the fourth and fifth equivalences hold by the definition of the Choquet integral and because $u(x) > 0$ by the assumption that $x \succ y_*$ (note that $u(y_*) = 0$); and the implication holds by the weak additivity of θ by Scott's Theorem (Theorem 1) because we are now assuming that θ can be decomposed into the unique strictly increasing distortion function and a unique convex-ranged probability charge.

Step 6 (Completion of the proof). Because Steps 4 and 5 show that θ is convex-ranged and weakly additive, Scott's Theorem (Theorem 1) provides a sufficient and necessary condition for θ to be strongly decomposable. Together with Step 1, the whole proof of the statement that Axioms A1-A7 characterize RDSEU is now complete. □

A.2 Proof of Proposition 4

Proof of (i). Suppose that $\succsim$ is the CEU with a convex-ranged probability capacity θ. Also, suppose that $\succsim$ restricted on F_0 is the PS, too, with a convex-ranged probability charge μ.

For any $E, F \in 2^S$ and any $x, z \in X$ with $x \succ z$, define the binary savage acts $f := f^{E^*}$ and $g := f^{F^*}$ by

$$f^{E^*} := \begin{bmatrix} x & \text{on} & E \\ z & \text{on} & S \backslash E \end{bmatrix} \quad \text{and} \quad f^{F^*} := \begin{bmatrix} x & \text{on} & F \\ z & \text{on} & S \backslash F \end{bmatrix}.$$

Then, by definition, $p_{f,\mu}(\{x\}) = \mu(E)$ and $p_{g,\mu}(\{x\}) = \mu(F)$. Therefore, $p_{f,\mu}$ strictly stochastically dominates $p_{g,\mu}$ if and only if $\mu(E) > \mu(F)$ because $x \succ z$ and $\mu(S\backslash E) < \mu(S\backslash F)$ by the finite additivity of μ. Because the PS respects the ordering by strict stochastic dominance, we must have $f^{E^*} \succ f^{F^*} \Leftrightarrow \mu(E) > \mu(F)$.

On the other hand, because $\succsim$ is also represented by the CEU with the probability capacity θ, it follows that $f^{E^*} \succ f^{F^*} \Leftrightarrow (u(x) - u(z))\theta(E) + u(z) > (u(x) - u(z))\theta(F) + u(z)$ for some utility index u, which implies that $f^{E^*} \succ f^{F^*} \Leftrightarrow \theta(E) > \theta(F)$ because $u(x) - u(z) > 0$ by $x \succ z$.

Eventually, we have established that the convex-ranged probability charge μ and the convex-ranged probability capacity θ together satisfy

$$(\forall E, F \in 2^S) \quad \mu(E) > \mu(F) \Leftrightarrow \theta(E) > \theta(F)$$

where the convex-rangedness of μ and θ is a part of the assumptions. Therefore, by Nishimura and Ozaki (2017, A.1.3, p.256), θ can be strongly decomposable as $\theta = \gamma \circ \mu$, where γ is a strictly increasing distortion function, and thus, $\succsim$ is the RDSEU as desired. □

Proof of (ii). Take any simple savage act $f \in F_0$. Because $f(S)$ is finite, we may write f as

$$(\forall s \in S) \quad f(s) = \sum_{i=1}^{k} x_i E_i^*(s),$$

where $(\forall i)$ $x_i \in f(S) \subseteq X$, $E_i := f^{-1}(\{x_i\})$ and E_i^* denotes the indicator function of E_i as before. Obviously, $\langle E_i \rangle_{i=1}^k$ is a finite partition of S.

Now, assume that $\succsim$ is represented by the RDSEU with a probability capacity θ, a strictly increasing distortion function γ as well as a utility index u. Because $f(S)$ is finite and u is unique up to a positive affine transformation, we may assume that all $u(x_i)$'s are non-negative by applying such a transformation. Furthermore, we may assume that $u(x_i)$ is strictly decreasing in i, by renumbering the indices i's if necessary. Therefore, we may invoke (2) to represent the

Choquet integral of f as:

$$\int_S u \circ f(s)\, d(\gamma \circ \mu)(s) = u(x_1)\gamma \circ \mu(E_1) + \sum_{j=2}^{k} u(x_j)\Big(\gamma \circ \mu\left(\cup_{i=1}^{j} E_i\right) - \gamma \circ \mu\left(\cup_{i=1}^{j-1} E_i\right)\Big). \quad (12)$$

In (12), let $(\forall i = 1, \ldots, k)$ $\nu(E_i) := \gamma \circ \mu(\cup_{i=1}^{j} E_i) - \gamma \circ \mu(\cup_{i=1}^{j-1} E_i)$ (let $\cup_{i=1}^{0} E_i := \phi$). Then, it follows that $\sum_{i=1}^{k} \nu(E_i) = 1$. Also, it holds that $(\forall i)$ $u^{-1}(\{u(x_i)\}) = x_i$. (Recall that we assumed that $u(x_i)$ is strictly decreasing in i, implying that $x_i \neq x_j$ if $i \neq j$.) Thus, $\langle p_{f,\nu}(\{u^{-1}(\{u(x_i)\})\})\rangle_{i=1}^{k}$ describes the complete distribution of outcomes on X induced by f via θ, γ and u. Furthermore, if two simple Savage acts f and g have the identical induced distribution on X, the value of the right-hand side of (12) coincide, by choosing the same u so that the ranges of both f and g are contained in the set of non-negative real numbers. We, therefore, conclude that the left-hand side of (12), that is, the RDSEU representation of a simple Savage act serves as the preference functional for the PS.

The remaining task for us is to show that the RDSEU representation is strictly monotonic in the sense described in Section 4.[21]

To this end, rewrite the left-hand side of (12) as follows (again by assuming the non-negativity of u without loss of generality):

$$\begin{aligned}
\int_S (u \circ f)(s) d(\gamma \circ \mu)(s) &= \int_S (u \circ f)(s)\, d\theta(s) \\
&= -\int_S -(u \circ f)(s)\, d\theta'(s) \\
&= -\int_{-\infty}^{0} (\theta'(\{s \,|\, -(u \circ f)(s) \geq z\}) - 1)\, dz \\
&= -\int_{-\infty}^{0} (1 - \gamma(1 - \mu(\{s \,|\, -(u \circ f)(s) \geq z\})) - 1)\, dz \\
&= \int_{-\infty}^{0} \gamma(1 - \mu(\{s \,|\, -(u \circ f)(s) \geq z\}))\, dz\,, \qquad (13)
\end{aligned}$$

where the first equality is definitional: $\theta := \gamma \circ \mu$; the second equality holds by the formula of the Choquet integral by the conjugate capacity (Nishimura and Ozaki, 2017, Proposition 2.4.1, p.39), where the *conjugate* probability capacity of θ is denoted and defined by $(\forall A \in \mathscr{A})$ $\theta'(A) := 1 - \theta(S \backslash A)$; the third equality holds by the definition of the Choquet integral, (1); the fourth equality holds by the definition of the conjugate (see right above); and the last equality is a simple manipulation.

[21] See Grant and Kajii (2005, p.8-9) for a similar discussion to the one below.

Next, let f and g be simple Savage acts, assume that u is non-negative without loss of generality and assume that f stochastically dominates g :

$$(\forall z \in \mathbb{R}_+) \quad \mu(\{s \in S | (u \circ f)(s) \leq z\}) \leq \mu(\{s \in S | (u \circ g)(s) \leq z\}) .$$

Then, the following sequence of equivalent statements are obtained:

$$\begin{aligned}
& (\forall z \in \mathbb{R}_+) \ \ \mu(\{s \in S | (u \circ f)(s) \leq z\}) \leq \mu(\{s \in S | (u \circ g)(s) \leq z\}) \\
\Leftrightarrow \ & (\forall z \in \mathbb{R}_-) \ \ \mu(\{s \in S | (u \circ f)(s) \leq -z\}) \leq \mu(\{s \in S | (u \circ g)(s) \leq -z\}) \\
\Leftrightarrow \ & (\forall z \in \mathbb{R}_-) \ \ \mu(\{s \in S | -(u \circ f)(s) \geq z\}) \leq \mu(\{s \in S | -(u \circ g)(s) \geq z\}) \\
\Leftrightarrow \ & (\forall z \in \mathbb{R}_-) \ \ 1 - \mu(\{s \in S | -(u \circ f)(s) \geq z\}) \geq 1 - \mu(\{s \in S | -(u \circ g)(s) \geq z\}) \\
\Leftrightarrow \ & (\forall z \in \mathbb{R}_-) \ \ \gamma(1 - \mu(\{s \in S | -(u \circ f)(s) \geq z\})) \geq \gamma(1 - \mu(\{s \in S | -(u \circ g)(s) \geq z\})) ,
\end{aligned}$$

where the last equivalence holds by the strict increase of γ.

When f *strictly* stochastically dominates g, the weak inequalities in the above equivalent statements hold with the strict inequalities for some z. However, f and g are simple; the range of such z's has a positive Lebesgue measure.

Combining all these facts with the equation (13) proves that the preference functional is strictly monotonic, which completes the proof. □

Proof of (iii). Suppose that the binary relation $\succsim$ is represented by the CEU with the probability capacity θ as well as a utility index u, and it is also represented by the PS with a probability charge μ.

Suppose that $(\forall x, y \in X)\ x \sim y$ (as a pair of the degenerate constant acts). Then, (ii) trivially holds true because the representation of any simple lottery act $f \in L_0$ is given by $u(x)$ for any $x \in X$, which is trivially the SEU. We thus assume that there are two outcomes $x, y \in X$ such that $x \nsim y$.

Next, consider the two simple, in fact, binary lotteries $q, r \in Y$ defined by $q := (x, 1/2; y, 1/2)$ and $r := (x, 2/3; y, 1/3)$, where we may assume that $x \succ y$ without loss of generality by the previous paragraph. Then, the affinity of u shows that $u(q) = u(x)/2 + u(y)/2 > 2u(x)/3 + u(y)/3 = u(r)$. Therefore, we may assume the existence of such a pair of binary lotteries that $u(q) > u(r)$ holds true again without loss of generality.

Now, let $p, q, r \in Y$ be three binary lotteries satisfying that $\mathrm{supp}(p) = \mathrm{supp}(q) = \mathrm{supp}(r) := \{x, y\} \subseteq X$. By the preceding two paragraphs, we may assume that $x, y \in X$ is such that $x \nsim y$, and that $q, r \in Y$ is such that $q \succ r$. Also, let $A \in \mathscr{A}$ and consider the following two simple lottery acts f and g : $(\forall s \in S)\ f(s) := p$ and

$$g := \begin{bmatrix} q & \text{on} & A \\ r & \text{on} & S \backslash A \end{bmatrix}.$$

Here, we make two simple observations. First, note that the distributions induced on X by f and g, *i.e.,* $p_{f,\mu}$ and $p_{g,\mu}$, are equal if and only if

$$p(x) = \mu(A)q(x) + (1 - \mu(A))r(x) \tag{14}$$

by (3). Therefore, it holds that $f \sim g$ whenever (14) is satisfied because $\succsim$ is represented by the PS with μ.

Second, by the assumption that $u(q) > u(r)$, the CEU representations of f and g, denoted by $U(f)$ and $U(g)$, respectively, is given by:

$$\begin{aligned} U(f) &= p(x)u(x) + p(y)u(y) = p(x)u(x) + (1 - p(x))u(y) \text{ and} \\ U(g) &= u(q)\theta(A) + u(r)(1 - \theta(A)) \\ &= \Big(q(x)u(x) + q(y)u(y)\Big)\theta(A) + \Big(r(x)u(x) + r(y)u(y)\Big)(1 - \theta(A)) \\ &= \Big(\theta(A)q(x) + (1 - \theta(A))r(x)\Big)u(x) + \Big(\theta(A)q(y) + (1 - \theta(A))r(y)\Big)u(y) \\ &= \Big(\theta(A)q(x) + (1 - \theta(A))r(x)\Big)u(x) + \Big(\theta(A)(1 - q(x)) + (1 - \theta(A))(1 - r(x))\Big)u(y) \\ &= \Big(\theta(A)q(x) + (1 - \theta(A))r(x)\Big)u(x) + \Big(1 - \theta(A)q(x) - (1 - \theta(A))r(x)\Big)u(y), \end{aligned}$$

implying that

$$\begin{aligned} U(f) = U(g) \quad &\Leftrightarrow \quad \Big(p(x) - (\theta(A)q(x) + (1 - \theta(A))r(x))\Big)\Big(u(x) - u(y)\Big) = 0 \\ &\Leftrightarrow \quad p(x) = \theta(A)q(x) + (1 - \theta(A))r(x), \end{aligned} \tag{15}$$

where the second equivalence relation holds by the assumption that $u(x) \neq u(y)$.

Finally, by combining the equations (14) and (15), we have the following successive implications:

$$\begin{aligned} & \mu(A)q(x) + (1 - \mu(A))r(x) = \theta(A)q(x) + (1 - \theta(A))r(x) \\ \Rightarrow \quad & (\mu(A) - \theta(A))(q(x) - r(x)) = 0 \\ \Rightarrow \quad & \mu(A) = \theta(A), \end{aligned}$$

where the last implication holds by the fact that $q(x) \neq r(x)$ because $q \succ r$ is assumed. Because the last equality holds for an arbitrary $A \in \mathscr{A}$, we conclude that $\theta = \mu$, and thus, that $\succsim$ is represented by the SEU with the probability charge μ, which completes the proof. □

A.3 Details of Example 1

We show that the preference relation represented defined in Example 1 is represented by the CEU on the Savage act and satisfies PS for any AA-acts.

Step 1 (Showing that the preference defined in the main text is monotonic, and hence, it is PS.) By the way of enumerating x_j's and because γ is strictly increasing, V thus defined is monotonic with respect to the stochastic dominance relation between any two AA acts. Therefore, the preference with V defined by (6) is certainly the PS preference with the AA acts.

Step 2 (Showing that the restriction of (6) to the Savage acts is represented by the CEU.) First, recall that, when the relevant acts are only the Savage ones, (3) will become

$$(\forall x \in X) \quad p_{f,\mu}(x) = \mu\left(f^{-1}(\{x\})\right).$$

Therefore, the preference defined via (6) in the previous step will be simplified as follows: For any Savage act $f \in F_0$,

$$\begin{aligned}
V(p_{f,\mu}) &= u(x_1) - \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\, \gamma \left(\sum_{j=k+1}^{n} \mu\left(f^{-1}\left(\{x_j\}\right)\right)\right) \\
&= u(x_1) - \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\, \gamma \left(\sum_{j=k+1}^{n} \mu\left(\left\{s \in S \,\middle|\, u(f(s)) = u(x_j)\right\}\right)\right) \\
&= u(x_1) - \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\, \gamma\left(\mu\left(\{s \in S \mid u(f(s)) \leq u(x_{k+1})\}\right)\right) \\
&= u(x_1) - \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\, \gamma\left(1 - \mu\left(\{s \in S \mid u(f(s)) > u(x_{k+1})\}\right)\right) \\
&= u(x_1) + \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\left[1 - \gamma\left(1 - \mu\left(\{s \in S \mid u(f(s)) > u(x_{k+1})\}\right)\right)\right] \\
&\quad - \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1})) \\
&= \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\left[1 - \gamma\left(1 - \mu(\{s \in S \mid u(f(s)) > u(x_{k+1})\})\right)\right] + u(x_n) \\
&= \sum_{k=1}^{n-1} (u(x_k) - u(x_{k+1}))\, \theta\left(\{s \in S \mid u(f(s)) > u(x_{k+1})\}\right) + u(x_n) \\
&= \int_S u(f(s))\, d\theta(s),
\end{aligned}$$

where the last expression is the Choquet integral with respect to the probability capacity defined by

$$(\forall E \in \mathscr{A}) \quad \theta(E) := 1 - \gamma(1 - \mu(E)) \,.$$

Therefore, we conclude that the PS preference with the AA acts constructed in Step 1 will be represented by the CEU when it is restricted to the Savage acts.

Moreover, if we define a mapping $\tilde{\gamma} : [0, 1] \to [0, 1]$ by $(\forall x) \quad \tilde{\gamma}(x) := 1 - \gamma(1 - x)$, the capacity θ derived above can be decomposed into a probability charge μ and a strictly increasing surjective function $\tilde{\gamma}$ as $\theta = \tilde{\gamma} \circ \mu$, which reveals that the example of the PS preference with the AA acts will become the *rank-dependent subjective expected utility* (RDSEU), which is a proper subclass of the CEU preference when restricted to the Savage acts.

Step 3 (Completion of the proof.) We have offered an example that satisfies all the assumptions. Now let γ be any strictly increasing function from $[0, 1]$ onto itself, which is *not* the identity mapping. It is then obvious that the example we offered is not the SEU, which completes the proof.

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