Equilibrium in Continuous-Time Financial Markets: Endogenously Complete Markets

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Paper closely related to Duffie-Zame (1989). Models are not identical; ours more general in some respects, theirs more general in others.

Common Features of the Models:

1. Uncertainty specified in terms of a K-dimensional Brownian Motion $\beta(t, \omega)$.
2. $J = K + 1$ securities traded in continuous time; one is usually a deterministic "bond."
3. Securities characterized by their dividends.
4. Agents characterized by their endowments and utility functions.

5. A K-dimensional standard Brownian motion is a K-dimensional stochastic process B such that

   (a) $B(\omega, 0) = 0$ almost surely (i.e. $P(\{\omega : B(\omega, 0) = 0\}) = 1$)

   (b) *Continuity*: $B(\omega, \cdot)$ is continuous almost surely.

   (c) *Independent Increments*: If $0 \leq t_0 < t_1 < \cdots < t_m \in \mathcal{T}$,

   $$\{B(\cdot, t_1) - B(\cdot, t_0), \ldots, B(\cdot, t_m) - B(\cdot, t_{m-1})\}$$

   is an independent family of random variables.

   (d) *Normality*: If $0 \leq s \leq t$, $B(\cdot, t) - B(\cdot, s)$ is normal with mean $0 \in \mathbf{R}^K$ and covariance matrix $(t - s)I$, where I is the $K \times K$ identity matrix.

6. **Goal:** *Prove existence of equilibrium and determine its properties.*

7. What did Duffie and Zame do?

   (a) Verify that an Arrow-Debreu equilibrium exists. Induce securities prices from the Arrow-Debreu prices.

   (b) *Under the endogenous assumption that the induced securities prices satisfy a dynamic spanning condition* (the $(K+1) \times K$ Jacobian of securities prices with respect to the Brownian motion components has rank K for almost all (t, ω)), securities are dynamically complete (can replicate any Arrow-Debreu security by continuous trading of given securities).

   (c) Show that dynamic completeness implies that the securities prices induced by the Arrow are *equilibrium* prices in the *securities* market.

8. Obvious approach: Show that for a generic (open dense, residual, or relatively prevalent) set of primitives, the spanning condition holds for the endogenously detemined Arrow-Debreu equilibrium prices.

   (a) Nobody's done it.

   (b) We tried and failed.

9. **Approach** *Think about Brownian Motion as a random walk, and take limits.* This is essentially what we do.

   (a) We prove existence of a securities market equilibrium, in which markets are dynamically complete, without the endogenous assumption.

   (b) Theorem is universal, not generic.

      - The spanning condition needed for complete markets follows from the way information is revealed in the model.
      - Given a continuous-time model, we can *choose* discrete approximations from the generic set on which Duffie-Shafer and Magill-Shafer show equilibrium exists.

10. Suppose U is an open subset of $\mathbf{R}^K$. $f : U \to \mathbf{R}$ is *analytic* if, for every $x \in U$, f can be represented on a neighborhood of x by a power series with positive radius of convergence. Key Facts:

- If an analytic function f is zero on a set of positive measure, then f is identically zero.
- *Analytic Implicit Function Theorem*: In the usual statement of the Implicit Function Theorem, if the given function is analytic, the implicitly defined function is also analytic.

Our Model:

1. Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure ν which agrees with Lebesgue measure on $[0, T)$ and such that $\nu(\{T\}) = 1$.

2. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F}, \nu)$. There is a standard K-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)$ adapted to the filtration. Let $\mathcal{I}(t, \omega) = (t, \beta(t, \omega))$.

3. There are I agents $i = 1, \ldots, I$. The endowment of the agent i is a process

$$e_i(t, \omega) = \begin{cases} f_i(\mathcal{I}(t, \omega)) & \text{if } t \in [0, T) \\ F_i(\mathcal{I}(T, \omega)) & \text{if } t = T \end{cases}$$

where $f_i : [0, T] \times \mathbf{R}^K \to \mathbf{R}_{++}$ and $F_i : \{T\} \times \mathbf{R}^K \to \mathbf{R}_{++}$ are analytic. $e(t, \omega) = \Sigma_{i=1}^I e_i(t, \omega)$ is the aggregate endowment.

4. There are $J+1 = K+1$ tradable securities (indexed by $j = 0, \ldots, J$) which pay dividends

$$A_j(t,\omega) = \begin{cases} g_j(\mathcal{I}(t,\omega)) & \text{if } t \in [0,T) \\ G_j(\mathcal{I}(T,\omega)) & \text{if } t = T \end{cases}$$

where $g_j : [0,T] \times \mathbf{R}^K \to \mathbf{R}_+$ and $G_j : \{T\} \times \mathbf{R}^K \to \mathbf{R}_{++}$ are analytic functions. Net supply of security j is $\eta_j \in \{0,1\}$.

5.

$$\exists_{m>0}\; e(t,\omega) + \sum_{j=0}^{J} \eta_j A_j(t,\omega) \;\geq\; m$$

$$\exists_{r>0}\; e(t,\omega) + \sum_{j=0}^{J} A_j(t,\omega) \;\leq\; r + e^{r|\beta(t,\omega)|}$$

6. Nondegeneracy Condition: for some $\beta \in \mathbf{R}^K$, the $(K+1) \times K$ Jacobian matrix of G with respect to β has rank K.

7. Agent i is initially endowed with deterministic security holdings $e_{iA} = (e_{iA_0}, \ldots, e_{iA_J}) \in \mathbf{R}_+^{J+1}$ satisfying

$$\sum_{i=1}^{I} e_{iA_j} = \eta_j$$

The initial security holdings are assumed nonnegative; otherwise, an agent might never be able to make good on his/her initial short position, and hence no equilibrium would exist.

8. Utility function of agent i:

$$U_i(c) = E_\nu \left[\int_0^T h_i(c_i(t,\cdot), \beta(t,\cdot), t)dt + H_i(c_i(T,\cdot), \beta(T,\cdot)) \right]$$

where the functions $h_i : \mathbf{R}_+ \times \mathbf{R}^K \times [0,T] \to \mathbf{R} \cup \{-\infty\}$ and $H_i : \mathbf{R}_{++} \times \mathbf{R}^K \to \mathbf{R} \cup \{-\infty\}$ are analytic on $\mathbf{R}_{++} \times \mathbf{R}^K \times [0,T]$ and $\mathbf{R}_{++} \times \mathbf{R}^K$ respectively and satisfy

$$\begin{array}{lcll}
\lim_{c\to 0_+} \frac{\partial h_i}{\partial c} & = & \infty & \text{uniformly in } (\beta, t \\
\lim_{c\to 0_+} \frac{\partial H_i}{\partial c} & = & \infty & \text{uniformly in } \beta \\
\lim_{c\to \infty} \frac{\partial h_i}{\partial c} & = & 0 & \text{uniformly in } (\beta, t \\
\lim_{c\to \infty} \frac{\partial H_i}{\partial c} & = & 0 & \text{uniformly in } \beta \\
\lim_{c\to 0_+} h_i(c,\beta,t) & = & h_i(0,\beta,t) & \text{uniformly in } (\beta, t \\
\lim_{c\to 0_+} H_i(c,\beta) & = & H_i(0,\beta) & \text{uniformly in } \beta \\
\left.\frac{\partial h_i}{\partial c}\right|_{(c,\beta,t)} & > & 0 & \text{for } c \in \mathbf{R}_{++} \\
\left.\frac{\partial^2 h_i}{\partial c^2}\right|_{(c,\beta,t)} & < & 0 & \text{for } c \in \mathbf{R}_{++} \\
\forall_{c>0} \exists_{M\in\mathbf{R}} \forall_{(\beta,t)} \left.\frac{\partial h_i}{\partial c}\right|_{(c,\beta,t)} & \leq & M & \\
\forall_{c>0} \exists_{M\in\mathbf{R}} \forall_{\beta} \left.\frac{\partial H_i}{\partial c}\right|_{(c,\beta)} & \leq & M &
\end{array}$$

Conditions satisfied by all state-independent CARA, CRRA utility functions.

9. (a) Consumption price process is $p_C(t,\omega) \in L^\infty([0,T] \times \Omega)$.

(b) Securities price process is Itô process $p_A = (p_{A_0}, \ldots, p_{A_J}) : \Omega \times [0,T] \to \mathbf{R}^{J+1}$ such that associated cumulative gains process

$$\gamma_j(t,\omega) = p_{A_j}(t,\omega) + \int_0^t p_C(s,\omega) A_j(s,\omega)\, ds$$

is a martingale. Securities are priced *cum dividend* at time T.

(c) Trading strategy for agent i:

- $z_i \in \mathcal{L}^2(\gamma)$, i.e. if the instantaneous volatility matrix of p_A is σ, then $z_i \cdot \sigma \in \mathcal{L}$, i.e. z_i is Itô integrable with respect to γ.
- $\int z_i d\gamma$ is a martingale (Harrison-Kreps admissibility condition).

10. Budget set for agent i is the set of all consumption plans c_i such that there exists trading strategy z_i such that c_i and t_i satisfy budget constraint

$$
\begin{aligned}
& p_A(t,\omega) \cdot z_i(t,\omega) \\
& \quad = e_{iA}(\omega) \cdot p_A(0,\omega) \\
& \qquad + \int_0^t z_i \, d\gamma + \int_0^t p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds \\
& \qquad \text{for almost all } \omega \text{ and all } t \in [0,T) \\
0 \; & = p_A(0,\omega) \cdot e_{iA}(0,\omega) + \int_0^T z_i \, d\gamma \\
& \qquad + \int_0^T p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds \\
& \qquad + p_C(T,\omega)(e_i(T,\omega) - c_i(T,\omega)) \\
& \qquad \text{for almost all } \omega
\end{aligned}
$$

11. Demand maximizes utility over the budget set.

12. Equilibrium: p_A, p_C, z_i, c_i in demand set so that markets clear: for almost all (t,ω)

$$
\begin{aligned}
\sum_{i=1}^{I} z_{iA_j}(t,\omega) &= \eta_j \text{ for } j = 0, \ldots, J \\
\sum_{i=1}^{I} c_i(t,\omega) &= \sum_{i=1}^{I} e_i(t,\omega) + \sum_{j=0}^{J} \eta_j A_j(t,\omega)
\end{aligned}
$$

Theorem 1 *The continuous-time finance model just described has an equilibrium, which is Pareto optimal.*

- *The equilibrium securities prices and consumption prices are given by analytic functions of* $\mathcal{I}(t, \omega)$ *for* $t \in [0, T)$*, and as (separate) analytic functions of* $\mathcal{I}(T, \omega)$ *for* $t = T$*.*
- *There is a closed set of measure zero in* $[0, T) \times \Omega$ *and an analytic function of* $\mathcal{I}(t, \omega)$ *defined on the complement of that set such that the equilibrium trading strategies equal this function on its domain.*
- *The equilibrium prices are effectively dynamically complete: any integrable consumption process which is adapted to the* Brownian *filtration can be replicated by a unique trading strategy.*
- *(Not Yet Formulated) The equilibria of discrete approximating models converge to*

equilibria of the continuous-time model.

Outline of Proof:

1. As in Raimondo (2005) (single agent model, with or without dynamic completeness) hyperfinite time axis is $\mathcal{T} = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$, where $\Delta T \simeq 0$ is an infinitesimal in nonstandard analysis.

2. If we used original nonstandard construction of Brownian motion (Anderson (1976)), each node would have 2^K successor nodes, ruling out dynamic completeness if $K > 1$. Instead, construct a hyperfinite random walk $\hat{\beta}$ in $\mathbf{R}^K$ such that each node has $K+1$ successor nodes and

$$\begin{aligned} E(\hat{\beta}(t + \Delta T, \cdot)|(t, \omega_0)) &= \hat{\beta}(t, \omega_0) \\ E(\Delta\hat{\beta}_i(t, \omega))(\Delta\hat{\beta}_j(t, \omega)) &= \frac{\delta_{ij}}{\Delta T} \end{aligned}$$

   Show that $\beta(t, \omega) = {}^\circ\hat{\beta}(\hat{t}, \omega)$ is a standard Brownian motion (not quite covered in the earlier papers in nonstandard probability).

3. Use the analytic functions to induce endowments, utility functions, and security payoffs in the hyperfinite economy.

4. Equilibrium: $\hat{p}_A$, $\hat{p}_C$, $\hat{z}_i$, $\hat{c}_i$ in demand sets so that markets clear: all $t \in \mathcal{T}$ and all $\omega \in \hat{\Omega}$

$$\begin{aligned} \sum_{i=1}^{I} \hat{z}_i(t,\omega) &= (\eta_0, \ldots, \eta_J) \\ \sum_{i=1}^{I} \hat{c}_i(t,\omega) &= \sum_{i=1}^{I} \hat{e}_i(t,\omega) + \sum_{j=0}^{J} \eta_j \hat{A}_j(\omega,t) \end{aligned}$$

5. Perturb endowments and security dividends by at most $(\Delta T)^2$ to ensure existence of Pareto optimal equilibrium with dynamically complete securities prices in hyperfinite economy (Duffie-Shafer (1985,1986), Magill-Shafer).
   - Does not rule out determinant being infinitesimal ("infinitesimal Hart points")
   - Infinitesimal Hart points in hyperfinite model will become Hart points in continuous time model
   - Need to show that infinitesimal Hart points are a set of Loeb measure zero

6. Marginal utility of consumption is infinite at zero, aggregate consumption is strictly positive at each node, so equilibrium consumptions of all agents strictly positive (possibly infinitesimal) at each node.

7. Let Δ be open $I-1$-dimensional simplex in $\mathbf{R}^I_{++}$. Pareto optimality implies there exists $\lambda = (\lambda_1, \ldots, \lambda_I) \in {}^*\Delta$ such that at each node (t, ω), there is a positive constant $\mu(t, \omega)$ such that

$$\lambda_1 {}^*\frac{\partial h_1}{\partial c}(\hat{c}_i(t,\omega), \hat{\beta}(t,\omega), t) = \cdots = \lambda_I {}^*\frac{\partial h_I}{\partial c}(\hat{c}_I(t,\omega), \hat{\beta}(t,\omega), t) = \mu(t,$$
$$\lambda_1 {}^*\frac{\partial H_1}{\partial c}(\hat{c}_i(\hat{T},\omega), \hat{\beta}(t,\omega)) = \cdots = \lambda_I {}^*\frac{\partial H_I}{\partial c}(\hat{c}_I(\hat{T},\omega), \hat{\beta}(t,\omega)) = \mu(\hat{T},$$

Let $\hat{c}(t,\omega) = \Sigma_{i=1}^I \hat{c}_i(t,\omega)$. By Analytic Implicit Function Theorem, there exist standard *analytic* functions

$$\begin{aligned}
\mu(t,\omega) &= {}^*\hat{\pi}((\lambda_1, \cdots, \lambda_I), \hat{c}(t,\omega), \hat{\beta}(t,\omega), t) \text{ for } t < \hat{T} \\
\mu(\hat{T},\omega) &= {}^*\hat{\Pi}((\lambda_1, \cdots, \lambda_I), \hat{c}(\hat{T},\omega), \hat{\beta}(\hat{T},\omega)) \\
\hat{c}_i(t,\omega) &= {}^*\hat{\psi}_i((\lambda_1, \cdots, \lambda_I), \hat{c}(t,\omega), \hat{\beta}(t,\omega), t) \text{ for } t < \hat{T} \\
\hat{c}_i(\hat{T},\omega) &= {}^*\hat{\Psi}_i((\lambda_1, \cdots, \lambda_I), \hat{c}(\hat{T},\omega), \hat{\beta}(\hat{T},\omega))
\end{aligned}$$

8. $\hat{p}_C(t,\omega) = \mu(t,\omega)$ are the Arrow-Debreu prices of consumption. Since total supply (from endowments and dividends) is uniformly bounded below, aggregate consumption uniformly bounded below, p_C is uniformly bounded above by a standard number.

9. First order conditions imply that $\hat{\gamma}_j$ is a hyperfinite martingale, and that $\hat{p}_{A_j}$ is given by the nonstandard extensions of standard analytic functions evaluated at $\hat{\lambda}, \hat{\beta}$, and the perturbed endowments and dividends.

10. There is a standard analytic function ρ such that

$$\hat{p}_A(t,\omega) = {}^*\rho(\lambda, \hat{\mathcal{I}}(t,\omega)) + O(\Delta T)$$

so define

$$\begin{aligned} p_A(t,\omega) &= {}^\circ\hat{p}_A(\hat{t},\omega) \\ &= \rho({}^\circ\lambda, \mathcal{I}(t,\omega)) \end{aligned}$$

is a standard analytic function of $\mathcal{I}(t,\omega)$.

11. The $(K+1)\times K$ Jacobian matrix of p_A with respect to the Brownian motion has rank K at one point (T,β); the entries are analytic, hence continuous, so the matrix has rank K on a set (t,β) of positive Lebesgue measure, hence the matrix has rank K everywhere except a closed set of Lebesgue measure zero.

12. Since the distribution of β is absolutely continuous with respect to Lebesgue measure, the set of (t,ω) such that the matrix has rank $< K$ is a set of measure zero: *that's the spanning condition!*

13. Form a continuous-time economy by applying Loeb measure construction to $\hat{\Omega}$. p_A, p_C, $c_1,\ldots,c_I$ is a candidate equilibrium for the Loeb economy. *Verify this is in fact an equilibrium: this process goes back to Brown-Robinson*

    (a) Since $\hat{p}_A$ is dynamically complete in the hyperfinite model, there is a unique nonanticipating matrix process $\hat{\sigma}$ such that $\hat{\gamma}(t,\omega) = \int_0^t \hat{\sigma}\,d\hat{\beta}$. $\hat{\sigma}$ is given by the nonstandard extension of a standard analytic function, so

$$\sigma(t,\omega) = {}^\circ\hat{\sigma}(t,\omega) = \Sigma(\mathcal{I}(t,\omega))$$

is a standard analytic function of $\mathcal{I}$. The cumulative gains process is

$$\begin{aligned}
\gamma(t,\omega) &= \int_0^t p_C(s,\omega)A(s,\omega)\,ds + p_A(t,\omega) \\
&= {}^\circ \int_0^{\hat{t}} \hat{p}_C(s,\omega)\hat{A}(s,\omega)\,ds + {}^\circ\hat{p}_A(t,\omega) \\
&= {}^\circ\hat{\gamma}(\hat{t},\omega) \\
&= {}^\circ \int_0^{\hat{t}} \hat{\sigma}\,d\hat{\beta} \\
&= \int_0^t {}^\circ\hat{\sigma}\,d\beta
\end{aligned}$$

(Anderson (1976), adapted to this hyperfinite random walk).

(b) z_i is Itô integrable with respect to γ and $\int z_i \, d\gamma$ is a martingale. $\hat{z}_i$ does not chatter because it is (except at the null set of infinitesimal Hart points) the nonstandard extension of an analytic function of $\hat{\mathcal{I}}$.

(c) c_i is in i's budget set (using the trading strategy z_i) because (Anderson (1976), adapted to this hyperfinite random walk)

$$^\circ \int_0^t \hat{z}_i \, d\hat{\gamma} = \int_0^{\circ t} z_i \, d\gamma$$

(d) Suppose $\bar{c}_i$ is in i's budget set, using trading strategy $\bar{z}_i$, and $\bar{c}_i$ yields higher utility. Since p_C is bounded above, markets are dynamically complete, and Inada conditions hold, we may assume that there is some $m > 0$ such that $\bar{c}_i(t, \omega) \geq m$ for all (t, ω). Since $\int \bar{z}_i \, d\gamma$ is a martingale, $\bar{c}_i \in L^1([0, T] \times \Omega)$ and

$$\int_{[0,T]\times\Omega} p_C \left(\bar{c}_i - e_i - \bar{z}_i A\right) = \bar{z}_i(0) p_A(0)$$

Lift $\bar{c}_i$ to an element $\hat{\bar{c}}_i \in SL^1(\mathcal{T}, \hat{\Omega})$, i.e. ${}^\circ\hat{\bar{c}}_i(t, \omega) = \bar{c}({}^\circ t, \omega)$ almost surely. We can arrange that $\hat{\bar{c}}_i(t, \omega) \geq m$ for all (t, ω); it follows that

$${}^\circ\hat{U}_i(\hat{\bar{c}}_i) = U_i(\bar{c}_i) > U_i(c_i) = {}^\circ\hat{U}_i(\hat{c}_i)$$

Because the hyperfinite market is dynamically complete, there is a unique hyperfinite trading strategy $\hat{\bar{z}}_i$ which produces the consumption stream $\hat{\bar{c}}_i$ so

$$\int_{\mathcal{T}\times\hat{\Omega}} \hat{p}_C \left(\hat{\bar{c}}_i - \hat{e}_i - \hat{\bar{z}}_i A\right) = \hat{\bar{z}}_i(0) \hat{p}_A(0)$$

$\hat{\bar{z}}_i(0)\hat{p}_A(0)$ might exceed $\hat{e}_A(0)\hat{p}_A(0)$; however, since the hyperfinite stochastic integral and the Itô integral agree up to an infinitesimal, the difference is infinitesimal. So we can reduce $\hat{\bar{c}}_i$ by an infinitesimal everywhere, increasing the security holdings (and hence the dividends received) everywhere, and to ensure that

$$\hat{\bar{z}}_i(0)\hat{p}_A(0) \leq \hat{e}_A(0)\hat{p}_A(0)$$

so the reduced $\hat{\bar{c}}_i$ lies in i's budget set. The reduction reduces utility by at most an infinitesimal, so we get a contradiction of the fact that $\hat{c}_i$ is in i's demand set. This contradiction shows that our candidate continuous-time equilibrium is in fact an equilibrium of the Loeb economy.

(e) Since the equilibrium prices, consumptions, and strategies in the Loeb economy are given by an analytic function of $\mathcal{I}$, we can use the same analytic functions to define prices, consumptions and strategies in the original economy. Since the Loeb economy has the same distribution as the orignal economy, the prices, consumptions and strategies form an equilibrium of the original economy.