Heterogeneous Risk Attitudes in a Continuous-Time Model

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Abstract

We prove that every continuous-time model in which all consumers have time-homogeneous and time-additive utility functions and share a common probabilistic belief and a common discount rate can be reduced to a static model. This result allows us to extend some of the existing results on the representative consumer and risk-sharing rules in static models to continuous-time models. We show that the equilibrium interest rate is lower and more volatile than in the standard representative consumer economy, and that the individual consumption growth rates are more dispersed than is predicted from the first-order conditions.

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1 Introduction

In this paper we consider a continuous-time model of asset markets populated by consumers with heterogeneous risk attitudes to explore implications of the heterogeneity onto asset pricing and efficient risk allocations. Although it is common in the analysis of asset markets to postulate a representative consumer with a utility function exhibiting constant relative risk aversion or, more generally, hyperbolic absolute risk aversion, we instead take an approach that is closer to reality, by explicitly modeling a group of heterogeneous consumers, and derive, rather than postulate, a utility function for the representative consumer. An important implication of this approach, which we shall establish in this paper, is that the equilibrium interest rate is lower and more volatile than is predicted by a representative consumer model of the above kind. We shall also obtain an interesting result on the degree of dispersion in the individual consumption growth rates arising from the heterogeneous risk attitudes.

The model of this paper is a simplest one of continuous time admitting heterogeneous risk attitudes. The uncertainty is described by a probability measure space Ω, and the gradual information revelation, along the time span $\boldsymbol{R}_+ = [0, \infty)$, is described by a filtration. They are

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assumed to be common across consumers, and hence the probabilistic belief is homogeneous and the information is always symmetric across all consumers. Every consumer has a time-separable and time-homogeneous expected utility function, by which we mean the utility functions are of the form $E\left(\int_0^\infty \exp(-\rho t) u(c_t)\, dt\right)$, with $c = (c_t)_{t \in \boldsymbol{R}_+}$ a stochastic consumption (rate) process and u sometimes referred to as the *Bernoulli* utility function. Thus the induced preference ordering is neutral with respect to the speed of information revelation, and there is no ambiguity aversion. The subjective time discount rate ρ is assumed to be common across consumers. Asset markets are assumed to be complete, so that all state-contingent future consumption processes are attainable through asset transactions.

The first result of this paper (Theorem 1) states that every such continuous-time model can be reduced to a static one by introducing an appropriate σ-field $\mathscr{M}$ on the product space $\Omega \times \boldsymbol{R}_+$ and defining an appropriate probability measure Q on $\mathscr{M}$. This can be done in such a way that all consumers have expected utility functions with respect to Q, which is common across them, the same Bernoulli utility function u appears in the new expected utility representation, and their preference orderings over consumption processes are unchanged.

In such a static model, there are some known results, reviewed in more detail in Section 4, on the representative consumer's utility function (constructed from the individual consumers' counterparts) and efficient risk-sharing rules. To present them here, let's first fix our terminology. For a Bernoulli utility function u and a consumption level x, the *absolute risk aversion* is $-u''(x)/u'(x)$. Its reciprocal, $-u'(x)/u''(x)$, is the *absolute risk tolerance*, denoted by $s(x)$. Its derivative with respect to the consumption level x, $s'(x)$, is the *absolute cautiousness*. It is easy to see that the absolute cautiousness is constant if and only if the absolute risk tolerance is linear,[1] which, in turn, is equivalent to the absolute risk aversion being hyperbolic.[2]

The benchmark result in a static model is the *mutual fund theorem*, which says that if all consumers have a common constant absolute cautiousness, then the representative consumer has the same constant absolute cautiousness. In particular, the representative consumer exhibits hyperbolic absolute risk aversion, and thus the use of the representative consumer model who is postulated to exhibit hyperbolic absolute risk aversion is justified. The theorem also says (the claim after which the theorem is named) that every consumer's risk-sharing rule, which determines his state-contingent consumption levels, is a linear function of aggregate consumption level alone. The mutual fund theorem thus provides a sufficient condition under which the linearity of the representative consumer's absolute risk tolerance and also of the individual consumers' risk sharing rules is guaranteed.

Hara, Huang, and Kuzmics (2005) (henceforth HHK for short), on the other hand, looked into the case in which the assumption of the common constant absolute cautiousness is not met. In the special case where all consumers have the constant absolute cautiousness (but their levels are not equal), they found that the representative consumer exhibits strictly convex

[1] Throughout this paper, "linear" means "affine", allowing for nonzero intercept on the vertical axis on its graph.

[2] Throughout this paper, we shall use "constant absolute cautiousness", "linear absolute risk tolerance", and "hyperbolic absolute risk aversion" interchangeably.

absolute cautiousness, and that the higher an individual consumer's absolute cautiousness, the more convex his risk-sharing rule. HHK thereby identified the nature of nonlinearity of the representative consumer's absolute cautiousness and the individual consumers' risk-sharing rules. We will see in this paper that the reduction theorem (Theorem 1) then guarantees that both the mutual fund theorem and the results of HHK are valid in the continuous-time model as well.

Given the prevalent use of first-order conditions in economic theory, the study of nonlinearity may be perceived as only of secondary importance. Such perception is not warranted, however, in a continuous-time model. This is due to Ito's Lemma, which roughly states that if $X = (X_t)_{t\in \boldsymbol{R}_+}$ is an stochastic (Ito) process and $Y = (Y_t)_{t\in \boldsymbol{R}_+}$ is the stochastic processes defined from X via $Y_t = F(X_t, t)$ for some two-variable function F, then the drift term of Y, representing the instantaneous expected change in its value per unit of time, depends not only on the first derivative $\partial F(X_t, t)/\partial x$ (and $\partial F(X_t, t)/\partial t$) but also on the second derivative $\partial^2 F(X_t, t)/\partial x^2$. Thus, in a continuous-time model, the nonlinearity has a nonnegligible impact on the rate of change per unit of time of the stochastic process it defines. We will see in Section 6 that the strict convexity of the representative consumer's absolute risk tolerances makes the equilibrium interest rate lower and more volatile than is predicted by a representative consumer model in which the representative consumer is assumed to exhibit hyperbolic absolute risk aversion. We will also see that the individual consumption growth rates are more dispersed across consumers than is predicted from the first-order condition of the efficient risk-sharing rules.

In Section 5, we will also present some results on the asymptotic behavior of the representative consumer's absolute cautiousness and of the consumption shares of consumers with differing levels of cautiousness. These results hold under much weaker conditions than the above-mentioned results on interest rates. In particular, the aggregate endowment process may have jumps under these assumptions.

There is a large body of literature on continuous-time models of asset markets. Duffie (2001) is one of the standard textbooks covering recent developments in the field. We shall rely much on it, following its notation closely and skipping the proofs of well known results. Among many contributions in the field, Dumas (1989) and Wang (1996) deserve special attention. They both dealt with economies of two consumers exhibiting constant relative risk aversion, with different levels of the constants. The economy of Wang (1996) is a pure exchange one, as in this paper, in which the aggregate consumption process must be equal to an exogenously given aggregate endowment process, while the economy of Dumas (1989) is a productive one, in which the aggregate consumption process is endogenously determined by investment decisions. The results in Sections 5 and 6 of this paper can be considered as generalizations of some results of Wang (1996) to the case of more than two consumers, more general utility functions, and more general endowment processes, although, unlike Wang, we do not obtain any explicit solution for consumptions or interest rates.

Risk attitudes are by no means the only ingredient of an asset market model for which the heterogeneity matters for asset pricing. Other aspects of consumer preferences of which the heterogeneity may well be significant for risk sharing and asset pricing are elasticity of intertemporal substitution, and subjective time discount rates, which was explored in Gollier and Zeckhauser (2005). The heterogeneity in beliefs were thoroughly investigated in Calvet, Grandmont, and Lemaire (1999), and Epstein and Miao (2003) took up the heterogeneity of ambiguity, in the sense of Knightian uncertainty, in a continuous-time model to tackle the home bias puzzles. As for income and wealth, Gollier (2001) investigated the impact of heterogeneity in wealth onto interest rates and risk premia, and Franke, Stapleton, and Subrahmanyam (1998) and Hara and Kuzmics (2005) explored the implications of the heterogeneity in uninsurable background risks onto the efficient risk allocations. The heterogeneity in the types of assets that consumers can trade is investigated in a continuous-time model by Basak and Cuoco (1998). These types of heterogeneity are interesting and important, but we will leave them aside to concentrate on implications obtained solely from the heterogeneity in risk attitudes.

In the macroeconomic literature, the most commonly used type of heterogeneity is the ex post heterogeneity. That is, the economy is populated by a continuum of ex ante homogeneous consumers, in the sense that they have the same expected utility function and initial endowments which are independently and identically distributed conditional on macroeconomic shocks. By the law of large numbers, the aggregate endowments are governed by macroeconomic shocks. The individual endowments are often assumed to be uninsurable, and the focus of research is to examine both qualitative and quantitative implications of this incomplete market assumption, as was done in Weil (1992) and Krusell and Smith (1998). We shall not pursue this line of research, as the qualitative analysis for the case of ex ante identical consumers tends to be too specialized to provide many insights for the general case.

This paper is organized as follows. The basic static and continuous-time models are introduced in Section 2. The theorem on how to reduce a continuous-time model to a static one is presented in Section 3. The existing results on the static model are reviewed in Section 4. We then present rather straightforward applications of these results to the continuous-time model in Section 5. In Section 6, we obtain the above-mentioned results on the level and volatility of interest rates and the individual consumption growth rates. Concluding remarks and suggestions on future research are given in Section 7.

2 Two Models

To clarify the nature and applicability of our first result (Theorem 1), we define a static model and a continuous-time model in this section. To start, we introduce two common ingredients, the Bernoulli utility functions and the set of the states of the world, of the two models.

There are I *consumers*, $i \in \{1, \ldots, I\}$. Consumer i has a *Bernoulli* (also known as *von-Neumann Morgenstern*) *utility function* $u_i : D_i \to \boldsymbol{R}$, where $D_i \in \mathscr{B}(\boldsymbol{R})$, that is, D_i is a Borel subset of $\boldsymbol{R}$,[3] representing the possible consumption levels for consumer i. We assume throughout that u_i is continuous.

[3]Throughout this paper, whenever we speak of measurable functions defined on or taking values in $\boldsymbol{R}$ or its subsets, we mean that $\boldsymbol{R}$ is endowed with the Borel σ-field $\mathscr{B}(\boldsymbol{R})$.

The uncertainty of the economy is described by a probability measure space $(\Omega, \mathscr{F}, P)$. The probability measure P specifies the common belief on the likelihood of states. Denote by E the expectation with respect to P.

These common ingredients can therefore be summarized by a profile of utility functions, $((D_1, u_1), \ldots, (D_I, u_I))$ and the state space $(\Omega, \mathscr{F}, P)$.

We do not explicitly describe assets, asset prices, or portfolios in either model. We assume that the markets are complete, which implies that an asset price process can be identified with a state-price deflator, and that every equilibrium allocation is Pareto efficient. Hence, to investigate the properties of the asset market equilibrium, it suffices to identify the properties of Pareto efficient allocations and their supporting state-price deflators, without referring to asset prices or dynamic asset trading strategies that implement the efficient allocations.

2.1 Static Model

There is no additional ingredient to fully define a static model but we need to specify the consumption set and the preference relation for each consumer.

Denote by $\mathscr{L}^0(\Omega, \mathscr{F}, P)$, or $\mathscr{L}^0(\Omega)$ for short, the set of all measurable random variables defined on the probability measure space Ω. For each consumer i, we define his *consumption set* Y_i as the set of all $c^i \in \mathscr{L}^0(\Omega)$ such that $c^i \in D_i$ P-almost surely. Denote by $\mathscr{L}^1(\Omega, \mathscr{F}, P)$, or $\mathscr{L}^1(\Omega)$ for short, the set of all integrable random variables defined on the probability measure space Ω. Define Z_i as the set of all $c^i \in Y_i$ such that $u_i\left(c^i\right) \in \mathscr{L}^1(\Omega)$. Then Z_i is the set of random variables c^i for which the expected utility $E\left(u_i\left(c^i\right)\right)$ is well defined (finite). We shall refer to Z_i as the *effective consumption set*.

Define a binary relation $\succsim_i$ on Y_i by letting, for each $c^i \in Y_i$ and $b^i \in Y_i$, $c^i \succsim_i b^i$ if and only if either of the following two conditions is met: $b^i \notin Z_i$; or $c^i \in Z_i$, $b^i \in Z_i$, and $E\left(u_i\left(c^i\right)\right) \geq E\left(u_i\left(b^i\right)\right)$. Then $\succsim_i$ is reflexive and transitive. Denote its strict part by $\succ_i$ and symmetric part by $\sim_i$, then $c^i \succ_i b^i$ for every $c^i \in Z_i$ and every $b^i \in Y_i \setminus Z_i$, and $c^i \sim_i b^i$ for every $c^i \in Y_i \setminus Z_i$ and every $b^i \in Y_i \setminus Z_i$. Thus the random variables c^i for which $u_i\left(c^i\right)$ is not integrable are the least preferable ones according to $\succsim_i$.

This definition of $\succsim_i$ can be justified as follows. In most applications, D_i is a non-degenerate interval, u_i is a concave function, and all random variables under consideration are integrable, so that, by definition, $c^i \in Y_i$ if and only if $c^i \in \mathscr{L}^1(\Omega)$ and $c^i \in D_i$ almost surely. Then, for every $x_i \in \operatorname{int} D_i$, there exists a $\theta_i \in \boldsymbol{R}$ (which can be any number between the right and left derivatives of u_i at x_i) such that $u_i\left(c^i\right) \leq \theta_i\left(c^i - x_i\right) + u_i\left(x_i\right)$. If $c^i \in \mathscr{L}^1(\Omega)$, then $\theta_i\left(c^i - x^i\right) + u_i\left(x^i\right) \in \mathscr{L}^1(\Omega)$, and hence $u_i\left(c^i\right)^+ \in \mathscr{L}^1(\Omega)$, where $u_i\left(c^i\right)^+$ is the positive part of $u_i\left(c^i\right)$. Hence, for every $c^i \in Y_i$, $c^i \in Z_i$ if and only if $u_i\left(c^i\right)^- \in \mathscr{L}^1(\Omega)$, where $u_i\left(c^i\right)^-$ is the negative part of $u_i\left(c^i\right)$. Roughly speaking, this means that every $c^i \in Y_i \setminus Z_i$ incurs negative infinite utility, and hence the way we have defined $\succsim_i$, by which the non-integrable c^i's are the least preferable ones, is intuitively consistent with the expected utility calculation.

In finance, it is also common to impose an even more stringent condition on c^i, the square integrability condition. That is, denoting by $\mathscr{L}^2(\Omega, \mathscr{F}, P)$, or $\mathscr{L}^2(\Omega)$ for short, the set of all square-integrable random variables defined on the probability measure space Ω, we define Y_i as the set of all $c^i \in \mathscr{L}^2(\Omega)$ such that $c^i \in D_i$ P-almost surely. The advantage of this approach is that $\mathscr{L}^2(\Omega)$ is self dual, so that every continuous linear functional on it admits the Riesz representation, which would then lead to a state-price deflator. In this paper, to simplify the analysis, we shall use this approach only when stating the decentralizing property of the state price deflator in Lemma 2.

2.2 Continuous-Time Model

To define a continuous-time model, we need to introduce three new ingredients in addition to the utility functions and the state space. The first one is the *time span* $\boldsymbol{R}_+$, which represents the timings at which consumption can take place. The choice of $\boldsymbol{R}_+$ means that the model is of continuous time with infinite horizon. Other case of time span will be discussed in Section 3. The second one is the filtration $\boldsymbol{F} = (\mathscr{F}_t)_{t \in \boldsymbol{R}_+}$, which describes the way in which the consumers receives gradually new information regarding the uncertainty of Ω. The third one is a positive number ρ, which represents the continuously compounded subjective discount rate. In short, a continuous-time economy is defined as a profile of utility functions, $((D_1, u_1), \dots, (D_I, u_I))$, a state space $(\Omega, \mathscr{F}, P)$, a time span $\boldsymbol{R}_+$, a filtration $\boldsymbol{F}$, and a continuously compounded common interest rate ρ. Note that $\boldsymbol{F}$ and ρ are common across all consumers. This means that there is no asymmetric information at any point in time and, as we will soon see, all consumers exponentially discount future utilities at the same rate.

We now need to define the consumption set for each consumer. For this purpose, we introduce some notation in steps.

Definition 1

1. Let $\mathscr{K}^0((\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$, denoted also as $\mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$ for short, be the linear space of all real-valued progressively measurable processes with respect to the filtration $\boldsymbol{F}$, that is, the set of functions $c : \boldsymbol{R}_+ \times \Omega \to \boldsymbol{R}$ such that the restriction of ζ onto $[0, t] \times \Omega$ is $(\mathscr{B}([0, t]) \otimes \mathscr{F}_t)$-measurable for every $t \in \boldsymbol{R}_+$ for every $t \in \boldsymbol{R}_+$.

   For each $c \in \mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$, we write $c_t(\omega)$ for $c(t, \omega)$, where $t \in \boldsymbol{R}_+$ and $\omega \in \Omega$. Then $c_t : \Omega \to \boldsymbol{R}$ is an $\mathscr{F}_t$-measurable random variable.

2. Then let $\mathscr{K}^1((\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$, denoted also as $\mathscr{K}^1(\boldsymbol{R}_+ \times \Omega)$ for short be the set of all processes in $\mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$ such that for P-almost every $\omega \in \Omega$, the sample path $t \mapsto \exp(-\rho t) c_t(\omega)$ is integrable with respect to the Lebesgue measure on $\boldsymbol{R}_+$; and the random variable $\omega \mapsto \int_0^\infty \exp(-\rho t) c_t(\omega)\, dt$ is integrable with respect to P.

3. Let Y_i be the set of all $c^i \in \mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$ such that there is a subset H of $\boldsymbol{R}_+ \times \Omega$ such that $\chi_H \in \mathscr{K}^1(\boldsymbol{R}_+ \times \Omega)$, where χ_H is the indicator function of H,[4] $E\left(\int_0^\infty \exp(-\rho t) \chi_{Ht}\, dt\right) = 0$, and $\{(t, \omega) \in \boldsymbol{R}_+ \times \Omega \mid c_t^i(\omega) \notin D_i\} \subseteq H$.[5]

[4]Since the indicator function takes values 0 or 1, $\chi_H \in \mathscr{K}^1(\boldsymbol{R}_+ \times \Omega)$ if and only if $\chi_H \in \mathscr{L}^0(\boldsymbol{R}_+ \times \Omega)$

[5]Since $\exp(-\rho t) > 0$, $E\left(\int_0^\infty \exp(-\rho t) \chi_{Ht}\, dt\right) = 0$ if and only if $E\left(\int_0^\infty \chi_{Ht}\, dt\right) = 0$.

4. Let Z_i be the set of all $c^i \in Y_i$ such that $u_i(c^i) \in \mathscr{K}^1(\boldsymbol{R}_+ \times \Omega)$.

This definition can be explained as follows: The set $\mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$ consists of all progressively measurable processes. The progressive measurability is in general stronger than the adaptedness (the property that c_t is $\mathscr{F}_t$-measurable for every $t \in \boldsymbol{R}_+$), but often regarded as not too much so, because every adapted process has a progressively measurable modification (as stated as Proposition 1.12 of Chapter 1 of Karatzas and Shreve (1991), for example). The set Y_i is the *consumption set*, consisting of the consumption processes that provide the consumer with consumption levels in D_i, almost surely in the sense stipulated in part 3. According to this definition, the consumption level may be outside D_i with a positive probability as long as it does not last for any period of positive length. The set Z_i is the *effective consumption set*, consisting of the progressively measurable consumption processes with the finite expected utility $E\left(\int_0^\infty \exp(-\rho t) c_t^i \, dt\right)$. Note that all consumers' utility functions are assumed to satisfy, in addition to having the expected utility representation, the time separability and time homogeneity.

We define a binary relation $\succsim_i$ on Y_i, accommodating the possibility of infinite utility levels, by letting, for each $c^i \in Y_i$ and $b^i \in Y_i$, $c^i \succsim_i b^i$ if and only if either of the following two conditions is met: $b^i \notin Z_i$; or $c^i \in Z_i$, $b^i \in Z_i$, and $E\left(\int_0^\infty \exp(-\rho t) u_i\left(c_t^i\right) dt\right) \geq E\left(\int_0^\infty \exp(-\rho t) u_i\left(b_t^i\right) dt\right)$. The definition of this preference relation can be justified as in the static model.

In the continuous-time finance, it is also common to impose the square integrability condition. That is, denote by $\mathscr{K}^2\left((\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho\right)$, or $\mathscr{K}^2(\boldsymbol{R}_+ \times \Omega)$ for short, the set of all processes in $\mathscr{L}^0(\boldsymbol{R}_+ \times \Omega)$ such that for P-almost every $\omega \in \Omega$, the sample path $t \mapsto \exp(-\rho t)(c_t(\omega))^2$ is integrable with respect to the Lebesgue measure on $\boldsymbol{R}_+$; and the random variable $\omega \mapsto \int_0^\infty \exp(-\rho t)(c_t(\omega))^2 \, dt$ is integrable with respect to P. Then define Y_i as the set of all $c^i \in \mathscr{K}^2(\boldsymbol{R} \times \Omega)$ such that $c^i \in D_i$ almost surely, in the sense of part 3 of Definition 1. Again, the advantage of this approach is that every continuous linear functional on $\mathscr{K}^2(\boldsymbol{R} \times \Omega)$ admits the Riesz representation, which would then lead to a state-price deflator. In this paper, to simplify the analysis, we shall use this approach only when stating the decentralizing property of the state price deflator in Lemma 4.

3 Reduction Theorem

Our first result asserts that every continuous-time model can be reduced to a static model. It draws heavily on Section 1.5 of Chung (1980).

Theorem 1 *Let* $(((D_1, u_1), \ldots, (D_I, u_I)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$ *be a continuous-time model. Then:*

1. *There exist a* σ*-field* $\mathscr{M}$ *on* $\boldsymbol{R}_+ \times \Omega$ *and a probability measure* Q *on* $(\boldsymbol{R}_+ \times \Omega, \mathscr{M})$ *such*

that $\mathscr{L}^1(\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q) = \mathscr{K}^1\left((\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho\right)$ *and*

$$E\left(\int_0^\infty \exp(-\rho t) c_t^i \, dt\right) = \rho E^Q(c) \tag{1}$$

for every $c \in \mathscr{K}^1\left((\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho\right)$.

2. *If the profile* $((Y_1, Z_1, \succsim_1), \dots, (Y_I, Z_I, \succsim_I))$ *of consumption sets, effective consumption sets, and preference relations corresponds to the above continuous-time model and the profile* $\left((\hat{Z}_1, \hat{Y}_1, \hat{\succsim}_1), \dots, (\hat{Y}_I, \hat{Z}_I, \hat{\succsim}_I)\right)$ *corresponds to the static model* $(((D_1, u_1), \dots, (D_I, u_I)), (\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q))$, *then* $Y_i = \hat{Y}_i$, $Z_i = \hat{Z}_i$, *and* $\succsim_i = \hat{\succsim}_i$ *for every* i.

This theorem asserts that a continuous-time model can be reduced to a static model, with the state space $\boldsymbol{R}_+ \times \Omega$, without affecting consumption sets or preference relations of any consumer.

Proof of Theorem 1 Following Section 1.5 of Chung (1980), we let $\mathscr{M}$ be the set of all subsets H of $\boldsymbol{R}_+ \times \Omega$ such that $H \cap ([0,t] \times \Omega) \in \mathscr{B}([0,t]) \otimes \mathscr{F}_t$ for every $t \in \boldsymbol{R}_+$. It is easy to check that $\mathscr{M}$ is indeed a σ-field on $\boldsymbol{R}_+ \times \Omega$.

By definition, a function $c : \boldsymbol{R}_+ \times \Omega \to \boldsymbol{R}$ is progressively measurable if and only if $(\bar{c}_t)^{-1}(B) \in \mathscr{B}([0,t]) \otimes \mathscr{F}_t$ for every $B \in \mathscr{B}(\boldsymbol{R})$ and every $t \in \boldsymbol{R}_+$, where $\bar{c}_t : [0,t] \times \Omega \to \boldsymbol{R}$ denotes the restriction of c on $[0,t] \times \Omega$. But $(\bar{c}_t)^{-1}(B) = c^{-1}(B) \cap ([0,t] \times \Omega)$. The progressive measurability is thus equivalent to saying that $c^{-1}(B) \cap ([0,t] \times \Omega) \in \mathscr{F}_t$ for every $B \in \mathscr{B}(\boldsymbol{R})$ and every $t \in \boldsymbol{R}_+$, which is in turn equivalent to $c^{-1}(B) \in \mathscr{M}$ for every $B \in \mathscr{B}(\boldsymbol{R})$. Therefore, c is progressively measurable if and only if c is $\mathscr{M}$-measurable.

Let λ^ρ be the measure on $(\boldsymbol{R}_+, \mathscr{B}(\boldsymbol{R}_+))$ of which the density function with respect to the Lebesgue measure λ is $\rho \exp(-\rho t)$. Let Q be the restriction of the product measure $\lambda^\rho \otimes P$ onto $\mathscr{M}$. Then, for every $c \in \mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$, c is integrable with respect to $\lambda^\rho \otimes P$ if and only if it is integrable with respect to Q. Moreover, since $Q(\boldsymbol{R}_+ \times \Omega) = (\lambda^\rho \otimes P)(\boldsymbol{R}_+ \times \Omega) = \lambda^\rho(\boldsymbol{R}_+)P(\Omega) = 1$, Q is in fact a probability measure. Since

$$E\left(\int_0^\infty \exp(-\rho t) c_t \, dt\right) = \rho \int_\Omega \left(\int_{\boldsymbol{R}_+} c(t, \omega) \, d\lambda^\rho(t)\right) dP(\omega),$$

Fubini's theorem (Theorems 8.4 and 8.7 in Chapter VI of Lang (1993), for example) implies that for every $c \in \mathscr{K}^0(\boldsymbol{R}_+ \times \Omega)$, $c \in \mathscr{K}^1(\boldsymbol{R}_+ \times \Omega)$ if and only if $c \in \mathscr{L}^1(\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q)$ and, if so, then

$$E\left(\int_0^\infty \exp(-\rho t) c_t \, dt\right) = \rho E^Q(c). \tag{2}$$

The proof of part 1 is thus completed.

As for part 2, note that $H \in \mathscr{M}$ if and only if its indicator function χ_H is progressively measurable. Then the equality $Y_i = \hat{Y}_i$ follows from the equivalence between the $\mathscr{M}$-measurability of H and the progressive measurability of its indicator function χ_H, and the equivalence between the $(\lambda^\rho \otimes P)$-almost sureness and the Q-almost sureness. The equality $Z_i = \hat{Z}_i$ is obtained by applying $\mathscr{K}^1(\boldsymbol{R}_+ \times \Omega) = \mathscr{L}^1(\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q)$ to $u_i(c^i)$. Finally, the equivalence between $\precsim_i$ and $\hat{\precsim}_i$ follows from these two equalities and (1). ///

Remark 1 As shown in the proof, the σ-field $\mathscr{M}$ has the property that each process is progressively measurable if and only if it is $\mathscr{M}$-measurable. But $\mathscr{M}$ is not the only σ-field that can be used for our analysis. We could instead impose the measurability requirement with respect to the *optional* σ-field $\mathscr{O}$ or the *predictable* σ-field $\mathscr{P}$, defined in Chapter 2 of Chung and Williams (1990). It is shown in its Chapter 3 that $\mathscr{O}$ is the smallest σ-field containing all right-continuous adapted processes; and that $\mathscr{P}$ is the smallest σ-field containing all left-continuous adapted processes. It is further shown that $\mathscr{P} \subset \mathscr{O} \subset \mathscr{M}$. The σ-fields $\mathscr{O}$ and $\mathscr{P}$ are conceptually more appealing, because they are generated by the continuity requirements on sample paths, and thus admit a natural interpretation in the analysis of intertemporal consumption patterns. We have, however, opted for $\mathscr{M}$, because the measurability with respect to it is a weaker requirement than the measurability with respect to $\mathscr{O}$ or $\mathscr{P}$, and hence the analysis based on $\mathscr{M}$ is more general.

Remark 2 While the time span is assumed to be of infinite horizon in Theorem 1, the result holds for any finite horizon $[0, T]$, with $0 < T < \infty$, as well. The only non-trivial modification we would need to make is to use the density function $\rho\,(1 - \exp(-\rho T))^{-1} \exp(-\rho t)$ to define the probability measure λ^ρ on $[0, T]$. Even if the consumer enjoys utility from wealth w^i at the terminal point T (which is measurable with respect to $\mathscr{F}_T$ and may be interpreted as a form of altruism) so that his utility function is

$$E\left(\int_0^T \exp(-\rho t) u_i(c_t^i)\, dt + \exp(-\rho T) u_i(w^i)\right),$$

the dynamic model can still be reduced to a static model with the state space $[0, T] \cup \{T^\dagger\}$, where $T^\dagger$ is an arbitrary point not in $[0, T]$, the σ-field consisting of the subsets of the form $M \cup (\{T^\dagger\} \times F)$ where $M \in \mathscr{M}$ and $F \in \mathscr{F}_T$, and the density function (Radon-Nikodym derivative) defined by

$$\frac{\rho}{1 - (1 - \rho)\exp(-\rho T)} \exp(-\rho t)$$

for the instantaneous consumption at $t \in [0, T]$ and

$$\frac{\rho}{1 - (1 - \rho)\exp(-\rho T)} \exp(-\rho T)$$

for the terminal wealth.

Remark 3 While the time span is assumed to be of continuous time in Theorem 1, the result holds for the time span of discrete time, $\{0, 1, 2, \dots\}$. The utility function $E\left(\int_0^\infty \exp(-\rho t) u_i(c_t^i)\, dt\right)$ would then be replaced by $E\left(\sum_{t=0}^\infty \delta_t u_i(c_t^i)\right)$, where $\delta = \exp(-\rho)$. In place of λ^ρ, we use the probability mass function (Radon-Nikodym derivative) $(1 - \delta)\delta_t$ for $t = 0, 1, 2, \dots$ (with respect to the counting measure on $\{0, 1, 2, \dots\}$). The finite horizon $\{0, 1, 2, \dots, T\}$ can be allowed for as well, in which case we should use the probability mass function (Radon-Nikodym derivative) $\dfrac{1-\delta}{1-\delta^{T+1}}\delta_t$ for $t = 0, 1, 2, \dots, T$ (with respect to the counting measure on $\{0, 1, 2, \dots, T\}$). The case of only finitely many periods and the case with terminal wealth could be accommodated with the minor modifications explained in Remark 2.

The case of discrete time is easier because all adapted processes are progressively measurable, and hence we do not need to be careful about the measurability requirement to be imposed on the consumption processes. In fact, the possibility to reduce a discrete-time model to a static one has been more or less well known, as documented in Section 4 of Chapter 15 of Gollier (2001).

4 Existing Results on the Static Model

We will see that our reduction theorem (Theorem 1) allows us to derive some properties of the efficient risk-sharing rules and the representative consumer's risk aversion in a continuous-time model from the corresponding results in the reduced static model. For this purpose, we review these results for the static model in this section. The materials here are either well known or contained in HHK.

We assume in the rest of this paper that for every consumer i, D_i is an open interval and u_i is infinitely many times differentiable and satisfies $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for every $x_i \in D_i$. We write $D_i = (\underline{d}_i, \overline{d}_i)$, where $\underline{d}_i \in \boldsymbol{R} \cup \{-\infty\}$, $\overline{d}_i \in \boldsymbol{R} \cup \{\infty\}$, and $\underline{d}_i < \overline{d}_i$. We also assume that u_i satisfies the *Inada condition*, that is, $u_i'(x_i) \to \infty$ as $x_i \to \underline{d}_i$ and $u_i'(x_i) \to 0$ as $x_i \to \overline{d}_i$.

Let $(((D_1, u_1), \dots, (D_I, u_I)), (\Omega, \mathscr{F}, P))$ be a static model. Let $((Y_1, Z_1, \succsim_1), \dots, (Y_I, Z_I, \succsim_I))$ be the corresponding profile of consumption sets, effective consumption sets, and preference relations, defined in Section 2.

Let $e \in \mathscr{L}^0(\Omega)$ be the aggregate endowment of the economy. A consumption allocation $\left(c^1, \dots, c^I\right) \in Y_1 \times \cdots \times Y_I$ is a *feasible* allocation of the aggregate endowment e if $\sum c^i = e$ P-almost surely. A feasible consumption allocation $\left(c^1, \dots, c^I\right) \in Y_1 \times \cdots \times Y_I$ is an *efficient* allocation (in the sense of Pareto) of the aggregate endowment e if there is no other feasible consumption allocation $\left(b^1, \dots, b^I\right) \in Y_1 \times \cdots \times Y_I$ of e such that $b^i \succsim_i c^i$ for every i, and $b^i \succ_i c^i$ for some i. While we shall not give a formal proof, it is easy to check that if there exists a feasible allocation $\left(b^1, \dots, b^I\right)$ of e such that $b^i \in Z_i$ for some i and if $\left(c^1, \dots, c^I\right)$ is an efficient allocation of e, then $c^i \in Z_i$ for *every* i. Under this assumption, therefore, an allocation is efficient if and only if it is efficient when the comparison is restricted to Z_i.

Write $D = \sum D_i$, which is an open interval of $\boldsymbol{R}$. For each $\lambda = (\lambda_1, \dots, \lambda_I) \in \boldsymbol{R}^I_{++}$ and each $x \in D$, consider the following maximization problem:

$$\begin{aligned} \max_{(x_1,\dots,x_I)\in D_1\times\cdots\times D_I} \quad & \sum \lambda_i u_i(x_i), \\ \text{subject to} \quad & \sum x_i = x. \end{aligned} \tag{3}$$

Under our assumptions, there exists a unique solution to this problem, which we denote by

$f_\lambda(x) = (f_{\lambda 1}(x), \dots, f_{\lambda I}(x))$ and the mapping $f_\lambda : D \to D_1 \times \cdots \times D_I$ is infinitely many times differentiable. The following lemma is well documented in the literature.

Lemma 1 *If $\left(c^1, \dots, c^I\right) \in Z_1 \times \cdots \times Z_I$ is an efficient allocation of the aggregate endowment e, then there exists a $\lambda \in \boldsymbol{R}^I_{++}$ such that $c^i = f_{\lambda i}(e)$ P-almost surely for every i. Conversely, for every $\lambda \in \boldsymbol{R}^I_{++}$, if $(f_{\lambda 1}(e), \dots, f_{\lambda I}(e)) \in Z_1 \times \cdots \times Z_I$, then it is an efficient allocation of e.*

By virtue of this lemma, for each $\lambda \in \boldsymbol{R}^I_{++}$, the mapping f_λ is called an *efficient risk-sharing rule.* By a slight abuse of terminology, we call each coordinate function $f_{\lambda i}$ an efficient risk-sharing rule (of consumer i) as well.

The first part of this lemma states that all efficient allocations for which all consumers' expected utilities are well defined can be fully characterized by a efficient risk-sharing rule for some utility weights $\lambda \in \boldsymbol{R}^I_{++}$. The second part establishes the converse of the first: The allocation generated by a efficient risk-sharing rule is an efficient allocation whenever all consumers' expected utilities are well defined. Without the assumption that $f_{\lambda i}(e) \in Z_i$ for every i, the generated allocation $(f_{\lambda 1}(e), \dots, f_{\lambda I}(e))$ need not be efficient. For this reason, the second part makes no claim on the existence of efficient allocations. Although we shall not elaborate on this point in this paper, there are, in fact, some cases where none of the feasible allocations is efficient.

Let f_λ be an efficient risk-sharing rule. Denote by $u_\lambda(x)$ the maximum attained in the problem (3). We are thereby defining a function $u_\lambda : D \to \boldsymbol{R}$, which is the value function of the problem. Since

$$\sum \lambda_i E\left(u_i(f_{\lambda i}(e))\right) = E\left(\sum \lambda_i u_i(f_{\lambda i}(e))\right) = E\left(u_\lambda(e)\right),$$

the function u_λ can be interpreted as the expected utility function of the representative consumer corresponding to the efficient risk-sharing rule f. Note that u_λ is infinitely many times differentiable. The following lemma states that its marginal utility provides a supporting state price deflator. Again, this is a well known result.

Lemma 2 *Let f_λ be an efficient risk-sharing rule and u_λ be its associated representative consumer's utility function. Suppose that $f_{\lambda i}(e) \in Z_i \cap \mathscr{L}^2$ for every i and $u'_\lambda(e) \in \mathscr{L}^2(\Omega)$. Then, for every i, $E(u'_\lambda(e)c^i) > E(u'_\lambda(e) f_{\lambda i}(e))$ whenever $c^i \in Z_i \cap \mathscr{L}^2(\Omega)$ and $c^i \succ_i f_{\lambda i}(e)$.*

In addition to the Inada condition, we now assume that every individual consumer's utility functions exhibit linear risk tolerance (or, equivalently, hyperbolic absolute risk aversion or constant absolute cautiousness). Specifically, we define the *absolute risk tolerance* s_i by $s_i(x_i) = -\dfrac{u'_i(x_i)}{u''_i(x_i)}$ for every $x_i \in D_i$ and assume that for every consumer i, there exist two real numbers κ_i and η_i such that

$$s_i(x_i) = \kappa_i x_i + \eta_i \tag{4}$$

for every $x_i \in D_i$. That is, the absolute risk tolerance, which is the reciprocal of the absolute risk aversion, is a linear function of consumption levels. The first derivative of the absolute risk tolerance, s_i' is the absolute cautiousness. The assumption (4) is equivalent to assuming that the absolute cautiousness is constantly equal to κ_i. The domain $D_i = (\underline{d}_i, \overline{d}_i)$ is determined so that the linear function is strictly positive on, and only on, D_i. It is then easy to check that depending on the sign of κ_i, η_i, $\underline{d}_i$ and $\overline{d}_i$ must be determined so that

$$\begin{aligned}\underline{d}_i = -\eta_i/\kappa_i > -\infty \text{ and } \overline{d}_i = \infty \text{ if } \kappa_i > 0,\\ \underline{d}_i = -\infty \text{ and } \overline{d}_i = \infty \text{ if } \kappa_i = 0,\\ \underline{d}_i = -\infty \text{ and } \overline{d}_i = -\eta_i/\kappa_i < \infty \text{ if } \kappa_i < 0,\end{aligned}$$

and

$$s_i(x_i) = \begin{cases} \kappa_i(x_i - \underline{d}_i) & \text{if } \kappa_i > 0, \\ \eta_i & \text{if } \kappa_i = 0, \\ -\kappa_i(\overline{d}_i - x_i) & \text{if } \kappa_i < 0. \end{cases}$$

The Inada condition is satisfied in all cases. Moreover, u_i exhibits constant relative risk aversion if and only if $\kappa_i > 0$ and $\eta_i = 0$, and then the constant equals $1/\kappa_i$. Denote by s_λ the absolute risk tolerance for the representative consumer's utility function u_λ. Write $\underline{d} = \sum \underline{d}_i$ and $\overline{d} = \sum \overline{d}_i$, then $D = (\underline{d}, \overline{d})$.

The mutual fund theorem is documented in, for example, Wilson (1968), Huang and Litzenberger (1988, Sections 5.15 and 5.26), Magill and Quinzii (1996, Section 3.16), Gollier (2001, Section 21.3.3), and LeRoy and Werner (2001, Section 15.6)) and can be stated in our notation as follows

Theorem 2 (Mutual Fund Theorem) *Let f_λ be an efficient risk-sharing rule and u_λ be the corresponding representative consumer's utility function. Suppose that (4) holds for every i and that $\kappa_1 = \cdots = \kappa_I$. Write $\kappa = \kappa_1 = \cdots = \kappa_I$ and $\eta = \eta_1 + \cdots + \eta_I$.*

1. *If $\kappa = 0$, then $s_\lambda(x) = \eta$ for every $x \in \boldsymbol{R}$, and there exist I numbers $n_1, \dots, n_i$ such that $\sum n_i = 0$ and $f_{\lambda i}(x) = \dfrac{\eta_i}{\eta} x + n_i$ for every i and $x \in \boldsymbol{R}$.*
2. *If $\kappa > 0$, then $s_\lambda(x) = \kappa(x - \underline{d})$ for every $x > \underline{d}$, and there exist I strictly positive numbers $m_1, \dots, m_i$ such that $\sum m_i = 1$ and $f_{\lambda i}(x) = m_i(x - \underline{d}) + \underline{d}_i$ for every i and $x > \underline{d}$.*
3. *If $\kappa < 0$, then $s_\lambda(x) = -\kappa(\overline{d} - x)$ for every $x < \overline{d}$, and there exist I strictly positive numbers $m_1, \dots, m_i$ such that $\sum m_i = 1$ and $f_{\lambda i}(x) = \overline{d}_i - m_i(\overline{d} - x)$ for every i and $x < \overline{d}$.*

Denote $\overline{\kappa} = \max_i \kappa_i$ and $\underline{\kappa} = \min_i \kappa_i$, and $\overline{I} = \{i \mid \kappa_i = \overline{\kappa}\}$ and $\underline{I} = \{i \mid \kappa_i = \underline{\kappa}\}$. Then $\kappa_1 = \cdots = \kappa_i$ if and only if $\overline{\kappa} = \underline{\kappa}$. The following result takes care of the case of $\overline{\kappa} > \underline{\kappa}$ and is established in Section 6 of HHK.

Theorem 3 (Hara, Huang, and Kuzmics (2005)) *Let f_λ be an efficient risk-sharing rule. Suppose that (4) hold for every i and that $\overline{\kappa} > \underline{\kappa}$.*

1. $s''_\lambda(x) > 0$ *for every* $x \in D$, $s'_\lambda(x) \to \overline{\kappa}$ *as* $x \to \overline{d}$, *and* $s'_\lambda(x) \to \underline{\kappa}$ *as* $x \to \underline{d}$.

2. $f''_{\lambda i}(x) > 0$ *for every* $i \in \overline{I}$ *and* $x \in D$.

3. $f''_{\lambda i}(x) < 0$ *for every* $i \in \underline{I}$ *and* $x \in D$.

4. *For every* $i \notin \overline{I} \cup \underline{I}$, *there exists a unique* $y_i \in D_i$ *such that* $f''_{\lambda i}(x) > 0$ *for every* $x < y_i$ *and* $f''_{\lambda i}(x) < 0$ *for every* $x > y_i$.

5. *For the* y_i *defined as in part 4,* $y_i < y_j$ *if* $\kappa_i < \kappa_j$; $y_i = y_j$ *if* $\kappa_i = \kappa_j$; *and* $y_i > y_j$ *if* $\kappa_i > \kappa_j$.

6. *If* $\underline{\kappa} > 0$, *then* $\sum_{i \in \overline{I}} \frac{f_i(x)}{x} \to 1$ *and* $\sum_{i \in \overline{I}} f'_{\lambda i}(x) \to 1$ *as* $x \to \infty$, *and* $\sum_{i \in \underline{I}} \frac{f_i(x) - \underline{d}_i}{x - \underline{d}} \to 1$ *and* $\sum_{i \in \underline{I}} f'_{\lambda i}(x) \to 1$ *as* $x \to \underline{d}$.

Part 1 of this theorem states that the slope of the representative consumer's risk tolerance is strictly increasing, from $\underline{\kappa}$ to $\overline{\kappa}$. This is equivalent to saying that the representative consumer's absolute cautiousness is a convex function of the aggregate consumption level. Part 2 states that the risk-sharing rule of a consumer with the largest absolute cautiousness is everywhere convex. Part 3 states that the risk-sharing rule of a consumer with the smallest absolute cautiousness is everywhere concave. Part 4 states that the risk-sharing rule of an intermediate consumer, who has neither the largest nor the smallest absolute cautiousness, is convex up to a (unique) inflection point, after which it is concave. Part 5 states that the larger the absolute cautiousness the higher the inflection point. Part 6 is concerned with the asymptotic properties of risk-sharing: For very high consumption levels, the consumers with the largest absolute cautiousness consume almost the entire aggregate consumption; and for very high consumption levels, close to the lower bound $\underline{d} > -\infty$, the dominant consumers are those with the smallest absolute cautiousness.

We should add that if appropriate specifications are made, then parts 1, 2, 3, and 6 of Theorem 3 remain to hold without the assumption (4). HHK contains these generations, and the subsequent analysis of this paper hold in the appropriate specifications without (4). It should therefore be considered as an assumption to simplify the exposition of this paper.

5 Asymptotic Properties in the Continuous-Time Model

We now turn Theorem 2 and parts of Theorem 3 to the corresponding results in the continuous-time model, by applying Theorem 1. Let $(((D_1, u_1), \ldots, (D_I, u_I)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$ be a continuous-time model, and $e \in \mathscr{L}^0(\boldsymbol{R}_+ \times \Omega)$ be the aggregate endowment process.

By Theorem 1, we can reduce the continuous-time model to a static model $(((D_1, u_1), \ldots, (D_I, u_I)), (\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q))$, while keeping the same consumption set Y_i, the effective consumption set Z_i, and the preference relation $\succsim_i$ for every i. The feasibility and efficiency of allocations of the consumption processes are also invariant under the reduction to the static model. Applying Lemmas 1 and 2 to the reduced static model, we obtain the following results on the continuous-time model $(((D_1, u_1), \dots, (D_I, u_I)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$.

Lemma 3 *If $\left(c^1, \dots, c^I\right) \in Z_1 \times \cdots \times Z_I$ is an efficient allocation of the aggregate endowment process e, then there exists a $\lambda \in \boldsymbol{R}^I_{++}$ such that $c^i = f_{\lambda i}(e)$ almost surely in the sense of part 3 of Definition 1 for every i. Conversely, for every $\lambda \in \boldsymbol{R}^I_{++}$, if $(f_{\lambda 1}(e), \dots, f_{\lambda I}(e)) \in Z_1 \times \cdots \times Z_I$, then it is an efficient allocation of e.*

Lemma 4 *Let f_λ be an efficient risk-sharing rule and u_λ be its associated representative consumer's utility function. Suppose that $f_{\lambda i}(e) \in Z_i \cap \mathscr{K}^2(\boldsymbol{R}_+ \times \Omega)$ for every i and $u'_\lambda(e) \in \mathscr{K}^2(\boldsymbol{R}_+ \times \Omega)$. Then, $(f_{\lambda 1}(e), \dots, f_{\lambda I}(e))$ is supported by the state price deflator $(\exp(-\rho t)u'(e_t))_{t \in \boldsymbol{R}_+}$ in the following sense: for every i,*

$$E\left(\int_0^\infty \exp(-\rho t) u'_\lambda(e_t)\, c^i_t \, dt\right) > E\left(\int_0^\infty \exp(-\rho t) u'_\lambda(e_t)\, f_{\lambda i}(e)\, dt\right)$$

whenever $c^i \in Y_i \cap \mathscr{K}^2(\boldsymbol{R}_+ \times \Omega)$ and $c^i \succ_i f_{\lambda i}(e_t)$.

As in the previous section, we impose the assumption of linear risk tolerance (4) for every i. We apply Theorems 2 and 3 to the static model static model $(((D_1, u_1), \dots, (D_i, u_i)), (\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q))$ to characterize the efficient allocations in the continuous-time model $(((D_1, u_1), \dots, (D_i, u_i)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$. The following result is the continuous-time version of the mutual fund theorem.

Proposition 1 *Let $(c^1, \dots, c^I)$ be an efficient allocation of the aggregate endowment process e in the continuous-time model $(((D_1, u_1), \dots, (D_I, u_I)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$. Suppose that (4) holds for every i and that $\kappa_1 = \cdots = \kappa_I$. Write $\kappa = \kappa_1 = \cdots = \kappa_I$ and $\eta = \eta_1 + \cdots + \eta_I$.*

1. *If $\kappa = 0$, then there exist I numbers $n_1, \dots, n_I$ such that $\sum n_i = 0$ and $c^i = \dfrac{\eta_i}{\eta} e + n_i$ almost surely in the sense of part 3 of Definition 1 for every i. Moreover, then, $(c^1, \dots, c^I)$ is supported by a state price deflator $(\exp(-\rho t)u'(e_t))_{t \in \boldsymbol{R}_+}$ in the sense of Lemma 4, where u_λ satisfies $s_\lambda(x) = \eta$ for every $x \in \boldsymbol{R}$.*

2. *If $\kappa > 0$, then there exist I strictly positive numbers $m_1, \dots, m_I$ such that $\sum m_i = 1$ and $c^i = m_i(e - \underline{d}) + \underline{d}_i$ almost surely in the sense of part 3 of Definition 1 for every i. Moreover, then, $(c^1, \dots, c^I)$ is supported by a state price deflator $(\exp(-\rho t)u'_\lambda(e_t))_{t \in \boldsymbol{R}_+}$ in the sense of Lemma 4, where u_λ satisfies $s_\lambda(x) = \kappa(x - \underline{d})$ for every $x > \underline{d}$.*

3. *If $\kappa < 0$, then there exist I strictly positive numbers $m_1, \dots, m_i$ such that $\sum m_i = 1$ and $c_i = \overline{d}_i - m_i(\overline{d} - \zeta)$ almost surely for in the sense of part 3 of Definition 1 every i. Moreover, then, $(c^1, \dots, c^I)$ is supported by a state price deflator $(\exp(-\rho t)u'_\lambda(e_t))_{t \in \boldsymbol{R}_+}$ in the sense of Lemma 4, where u_λ satisfies $s_\lambda(x) = -\kappa(\overline{d} - x)$ for every $x < \overline{d}$.*

Note in particular that the meaning of the mutual fund theorem is extended to this continuous-time setting: The mutual fund property is obtained not only of for risk-sharing at each $t \in \boldsymbol{R}_+$, but also intertemporally over the entire time span $\boldsymbol{R}_+$. However, the constituent assets of the two fund separation are different from the static model. In the static model, they are the risk-free discount bond, which pays one unit of the commodity almost surely at the single consumption period, and the market portfolio, which pays the aggregate endowment e at the single consumption period. In the continuous-time model, they are the perpetual bond, which pays one unit of the commodity almost surely and *continually at every* $t \in \boldsymbol{R}_+$ (rather than at some single future point in time), and the intertemporal market portfolio $e = (e_t)_{t\in\boldsymbol{R}_+}$, which pays the aggregate endowment e_t almost surely and *continually at every* $t \in \boldsymbol{R}_+$ (rather than at some single future point in time). If the perpetual bond were replaced by a discount bond maturing at some point in time, then the mutual fund theorem would in general fail. This phenomenon was investigated in details by Schmedders (2005).

We now turn to the consequence of applications of Theorem 3 to the reduced static model $(((D_1, u_1), \dots, (D_I, u_I)), (\boldsymbol{R}_+ \times \Omega, \mathscr{M}, Q))$ to identify some asymptotic properties of the efficient consumption allocations in the continuous-time model $(((D_1, u_1), \dots, (D_I, u_I)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$. The symbols, $\overline{\kappa}$, $\underline{\kappa}$, $\overline{I}$, and $\underline{I}$ are defined as before.

Proposition 2 *Let $(c^1, \dots, c^I)$ be an efficient allocation of the aggregate endowment process e in the continuous-time model $(((D_1, u_1), \dots, (D_i, u_i)), (\Omega, \mathscr{F}, P), \boldsymbol{R}_+, \boldsymbol{F}, \rho)$. Assume that (4) holds for every i.*

1. *If $e_t \to \infty$ P-almost surely as $t \to \infty$, then $s'_\lambda(e_t) \to \overline{\kappa}$ P-almost surely as $t \to \infty$. If $e_t \to \underline{d}$ P-almost surely as $t \to \infty$, then $s'_\lambda(e_t) \to \underline{\kappa}$ P-almost surely as $t \to \infty$.*
2. *Suppose that $\underline{\kappa} > 0$. If $e_t \to \infty$ P-almost surely, then $\sum_{i\in\overline{I}} \frac{c^i_t}{e_t} \to 1$ P-almost surely as $t \to \infty$, and if $e_t \to \underline{d}$ P-almost surely as $t \to \infty$, then $\sum_{i\in\underline{I}} \frac{c^i_t - \underline{d}_i}{e_t - \underline{d}} \to 1$ P-almost surely as $t \to \infty$.*

The first part of the above proposition establishes the asymptotic property of the absolute cautiousness of the representative consumer as the aggregate endowment tends to infinity or the minimum subsistence level $\underline{d}$. Specifically, it converges to the maximum individual absolute cautiousness $\overline{\kappa}$ as the aggregate endowment diverges to infinity, and it converges to the minimum individual absolute cautiousness $\underline{\kappa}$ as the aggregate endowment converges to the minimum subsistence level. The second part, on the other hand, deals with the asymptotic properties of the efficient risk-sharing rules. Specifically, the most absolutely cautious consumers take up almost all of the endowment as the it diverges to infinity, and the least absolutely cautious consumers take up almost all of the endowment, in excess of the minimum subsistence level, as it converges to the minimum subsistence level. Properties of this sort were also found in Section 3 of Wang (1996), but only in economies in which there are only two consumers exhibiting constant relative risk aversion and the aggregate endowment process is a geometric Brownian motion. The above proposition, in contrast, holds for economies of an arbitrary number of consumers having arbitrary utility functions exhibiting constant absolute cautiousness, and an arbitrary aggregate endowment process satisfying the required asymptotic properties, possibly with jumps. The fact that no stringent conditions are needed on sample paths is an advantage of our approach (which relies on the reduction theorem (Theorem 1)) over the approach relying on the stochastic optimal control.[6]

6 The Case of Stochastic Differential Equations

In the previous section, we applied the mutual fund theorem (Theorem 2) and a theorem in HHK (Theorem 3) to the reduced static model to obtain some properties of the efficient allocations and the state price deflator of the original continuous-time model. All these properties are asymptotic ones, in that they are concerned with the consumption shares and the representative consumer's absolute risk tolerance when the aggregate endowment level goes to zero or infinity. In this section, we assume that the aggregate endowment process is defined by a stochastic differential equation and investigate the implication of the heterogeneity in consumers' risk attitudes onto the interest rates and individual consumption growth rates.

Let $\lambda \in \boldsymbol{R}^I_{++}$, u_λ be the corresponding representative consumer's utility function, and $f_\lambda = (f_{\lambda 1}, \dots, f_{\lambda I})$ be the corresponding risk-sharing rule. Also, we assume that there are a one-dimensional Brownian motion $B = (B_t)_{t \in \boldsymbol{R}_+}$, generating the filtration $\boldsymbol{F}$, and two functions $\mu : D \times \boldsymbol{R}_+ \to \boldsymbol{R}$ and $\sigma : D \times \boldsymbol{R}_+ \to \boldsymbol{R}$ such that the endowment process $e = (e_t)_{t \in \boldsymbol{R}_+}$, taking values in D almost surely in the sense of part 3 of Definition 1, is a solution to the following stochastic differential equation:

$$de_t = \mu(e_t, t)\, dt + \sigma(e_t, t)\, dB_t.$$

The prime example of this type of the stochastic differential equation is where $D = \boldsymbol{R}_{++}$, $\mu(x,t) = \hat{\mu} x$, and $\sigma(x,t) = \hat{\sigma} x$ with some constants $\hat{\mu}$ and $\hat{\sigma}$, in which case the solution is a geometric Brownian motion

$$e_t = e_0 \exp\left(\left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right) t + \hat{\sigma} B_t\right), \tag{5}$$

with $e_0 > 0$. Then $e_t \to \infty$ almost surely as $t \to \infty$ if $\hat{\mu} > \hat{\sigma}/2$, and $e_t \to 0$ almost surely as $t \to \infty$ if $\hat{\mu} < \hat{\sigma}/2$. A general set of conditions on μ and σ guaranteeing the existence of a solution is given, for example, in Appendix E of Duffie (2001).

[6] This statement, in fact, should be taken with caution. With discontinuous sample paths, the progressive measurability of the endowment process is not automatically guaranteed by the adaptedness. For a given number of securities, the markets are less likely to be complete, because, for market completeness, there must be at least as many types of securities as possible levels of jumps. For more general endowment processes, therefore, the assumptions of a progressively measurable endowment process and complete markets are more likely to be significant restrictions.

6.1 Individual Consumption Growth Rates

For each i, define the consumption process c^i by $c^i = f_{\lambda i}(e)$. Then, by Ito's Lemma, c^i is an Ito process with

$$dc_t^i = \left(f'_{\lambda i}(e_t)\mu(e_t,t) + \frac{1}{2} f''_{\lambda i}(e_t) \left(\sigma(e_t,t)\right)^2 \right) dt + f'_{\lambda i}(e_t)\sigma(e_t,t)\, dB_t.$$

Let's denote the drift process by $\mu^i = (\mu_t^i)_{t\in \boldsymbol{R}_+}$:

$$\mu_t^i = f'_{\lambda i}(e_t)\mu(e_t,t) + \frac{1}{2} f''_{\lambda i}(e_t) \left(\sigma(e_t,t)\right)^2.$$

We are interested in the continuously compounded instantaneous expected rate of consumption growth of consumer i:

$$\lim_{\tau\downarrow t} \frac{1}{\tau - t} \log\left(\frac{E_t\left(c_\tau^i\right)}{c_t^i} \right) = \lim_{\tau\downarrow t} \frac{\log E_t\left(c_\tau^i\right) - \log c_t^i}{\tau - t} = \frac{d}{d\tau} \log E_t\left(c_\tau^i\right)\bigg|_{\tau=t}.$$

Since

$$\frac{d}{d\tau} E_t\left(c_\tau^i\right)\bigg|_{\tau=t} = \mu_t^i,$$

the chain rule differentiation implies that

$$\lim_{\tau\downarrow t} \frac{1}{\tau - t} \log\left(\frac{E_t\left(c_\tau^i\right)}{c_t^i} \right) = \frac{\mu_t^i}{c_t^i}. \tag{6}$$

That is, μ_t^i/c_t^i is the continuously compounded instantaneous expected rate of consumption growth of consumer i. Since $c_t^i = f_{\lambda i}(e_t)$,

$$\frac{\mu_t^i}{c_t^i} = \frac{f'_{\lambda i}(e_t)}{f_{\lambda i}(e_t)} \left(\mu(e_t,t) + \frac{1}{2} \left(\sigma(e_t,t)\right)^2 \frac{f''_{\lambda i}(e_t)}{f'_{\lambda i}(e_t)} \right). \tag{7}$$

This equality pinpoints the contribution in the continuous-time model of the curvature $f''_{\lambda i}(e_t^i)/f'_{\lambda i}(e_t^i)$ of the risk-sharing rules onto the conditionally expected instantaneous consumption growth rates. To appreciate its importance, note first that the multiplier $f'_{\lambda i}(e_t)/f_{\lambda i}(e_t)$ is what could be perceived as the first-order approximation of the consumption growth rate, as the numerator $f'_{\lambda i}(e_t)$ represents the infinitesimal change in the individual consumption level due to the infinitesimal increase in the aggregate consumption. Then (7) tells us that this perception is correct if the aggregate consumption growth is deterministic, but not so if it is stochastic. In the stochastic case, due to Ito's Lemma, the curvature of the risk-sharing rule contributes to the conditionally expected consumption growth rate, so that $f'_{\lambda i}(e_t)/f_{\lambda i}(e_t)$ is significantly different from the growth rate $(\tau - t)^{-1} \log\left(E_t\left(c_\tau^i\right)/c_t^i\right)$ even if $\tau - t$ is very close to zero and thus $E_t\left(c_\tau^i\right)$ is very close to c_t^i.

To see the nature of the bias generated by the curvature of the risk-sharing rule, note that by (3) and (8) of HHK,

$$
\begin{aligned}
\frac{\mu_t^i}{c_t^i} &= \frac{s_i\left(c_t^i\right)/s_\lambda(e_t)}{c_t^i}\left(\mu(e_t,t)+\frac{1}{2}\left(\sigma(e_t,t)\right)^2\frac{1}{s_\lambda(e_t)}\left(s_i'\left(c_t^i\right)-s_\lambda'(e_t)\right)\right) \\
&= \frac{s_i\left(c_t^i\right)/c_t^i}{s_\lambda(e_t)/e_t}\left(\frac{\mu(e_t,t)}{e_t}+\frac{1}{2}\left(\frac{\sigma(e_t,t)}{e_t}\right)^2\frac{1}{s_\lambda(e_t)/e_t}\left(s_i'\left(c_t^i\right)-s_\lambda'(e_t)\right)\right). \qquad (8)
\end{aligned}
$$

This is a general formula relating the individual absolute risk tolerance to his conditionally expected consumption growth rate. To make it easier to grasp, assume now that every consumer exhibits constant relative risk aversion. This is equivalent to assuming that $\eta_i = 0$ in (4) for every i, and also equivalent to assuming that $-u_i''(x_i)x_i/u_i'(x_i) = 1/\kappa_i$ for every i and $x_i > 0$. Then (8) can be rewritten as

$$
\frac{\mu_t^i}{c_t^i} = \frac{\kappa_i}{s_\lambda(e_t)/e_t}\left(\frac{\mu(e_t,t)}{e_t}+\frac{1}{2}\left(\frac{\sigma(e_t,t)}{e_t}\right)^2\frac{1}{s_\lambda(e_t)/e_t}\left(\kappa_i-s_\lambda'(e_t)\right)\right). \qquad (9)
$$

By (5) of HHK, $s_\lambda'(e_t)$ is a weighted average of the $s_i'(c_t^i)$. It can be derived from (4) of HHK that $s_\lambda(e_t)/e_t$ is a weighted average of the $s_i(c_t^i)/c_t^i$. Hence both are weighted averages of the κ_i. Since $s_\lambda''(e_t) \geq 0$ by Theorem 5 of HHK and $\lim_{x\to 0} s_\lambda(x) = \sum_i \lim_{x_i\to 0} s_i(x_i) = 0$, we have $s_i'(e_t^i) \geq s_\lambda(e_t)/e_t$. Thus

$$
\underline{\kappa} \leq \frac{s_\lambda(e_t)}{e_t} \leq s_\lambda'(e_t) \leq \overline{\kappa}, \qquad (10)
$$

where $\underline{\kappa} = \min\{\kappa_1,\dots,\kappa_I\}$ and $\overline{\kappa} = \max\{\kappa_1,\dots,\kappa_I\}$. Assume now that there exist a $\underline{\mu} \in \boldsymbol{R}$ and a $\overline{\sigma} \in \boldsymbol{R}_{++}$ such that $\mu(x,t)/x \geq \underline{\mu}$ and $\sigma(x,t)/x \leq \overline{\sigma}$ for every $(x,t) \in \boldsymbol{R}_{++} \times \boldsymbol{R}_{+}$. This assumption is satisfied by the geometric Brownian motion (5) with $\underline{\mu} = \hat{\mu}$ and $\overline{\sigma} = \hat{\sigma}$. Then

$$
\frac{\mu(e_t,t)}{e_t}+\frac{1}{2}\left(\frac{\sigma(e_t,t)}{e_t}\right)^2\frac{1}{s_\lambda(e_t)/e_t}\left(\kappa_i-s_\lambda'(e_t)\right) \geq \underline{\mu}-\frac{\overline{\sigma}^2}{2}\left(\frac{\overline{\kappa}}{\underline{\kappa}}-1\right). \qquad (11)
$$

Moreover, all the inequalities so far hold as strict inequalities unless all the κ_i are identical. Thus, if

$$
\underline{\mu} \geq \frac{\overline{\sigma}^2}{2}\left(\frac{\overline{\kappa}}{\underline{\kappa}}-1\right),
$$

then for any two consumers i and j, if $\kappa_i > \kappa_j$, then

$$
\frac{\mu_t^i/c_t^i}{\mu_t^j/c_t^j} > \frac{\kappa_i}{\kappa_j}.
$$

That is, the conditionally expected instantaneous consumption growth rates are more dispersed than are predicted by the individual constant relative risk aversion. This inequality can also be written as

$$
\frac{\mu_t^i/c_t^i}{\kappa_i} > \frac{\mu_t^j/c_t^j}{\kappa_j},
$$

which says that

$$\frac{\mu_t^i/c_t^i}{\kappa_i}$$

is an increasing function of κ_i. That is, the distribution of the individual consumption growth rates is more dispersed than the distribution of the individual constant relative risk aversion in the sense of the monotone likelihood ratio condition.

The risk-sharing rules are often used to test whether observed intertemporal consumption paths under uncertainty constitute an efficient allocation. Ignoring the integrability conditions, we can see from Lemma 3 that the efficiency of a feasible allocation $(c^1, \ldots, c^I)$ is equivalent to the existence of a mapping $f_\lambda = (f_{\lambda 1}, \ldots, f_{\lambda I}) : D \to D_1 \times \cdots \times D_I$ such that for every i, $c^i = f_{\lambda i}(\sum_i c_i)$ almost surely in the sense of part 3 of Definition 1. Hence every test for efficiency is a test for the existence of such an f_λ.

It is easy to see that $f'_{\lambda i}(x) > 0$ for every i and $x \in D$.[7] This property can be paraphrased as the *comonotonicity*: For every i and j, $c_t^i > c_\tau^i$ if and only if $c_t^j > c_\tau^j$. It can be shown[8] that if for every comonotone feasible allocation, there exists a collection of utility functions $u_1, \ldots, u_I$ so that $(c^1, \ldots, c^I)$ is an efficient allocation with respect to $u_1, \ldots, u_I$. We can therefore conclude that if we do not impose any restriction on the utility functions beyond monotonicity and risk aversion, for any observed comonotone consumption allocation, we do not reject the hypothesis that the allocation is efficient.

This appears to be too lenient a test for efficiency. The existing literature therefore imposed additional conditions on utility function to derive more restrictions on risk-sharing rules, which we shall now present in our continuous-time model. Townsend (1994), Mace (1991), and Kohara, Ohtake, and Saito (2002) conducted tests for efficiency for the cases where all consumers have the same utility function that exhibits either constant absolute risk aversion or constant relative risk aversion. In the case of constant relative risk aversion (so that, $\kappa_1 = \cdots = \kappa_I$ and $\eta_1 = \cdots = \eta_I = 0$), by the mutual fund theorem, the representative consumer also has the same constant relative risk aversion , and hence (9) implies that $s_\lambda(e_t)/e_t = s'_\lambda(e_t) = \kappa_i$ and $\mu_t^i/c_t^i = 1$ almost surely for every i. That is, every individual consumer's conditionally expected consumption growth rate is equal to the conditionally expected growth rate for the aggregate consumption. This apparently provide a rather stringent test for efficiency; and the hypothesis that the observed consumption paths constitute an efficient allocation is often rejected in the literature. Ogaki and Zhang (2001) relaxed the assumption of the common constant relative risk aversion and only assumed that all consumers have the same utility function exhibiting constant positive cautiousness. That is, (4) is satisfied for every i, with $\kappa_1 = \cdots = \kappa_I > 0$,

[7]This follows, for example, from (3) of HHK.

[8]This result should be stated with some care. For example, Dana (2004, Proposition 10) proved it under the assumption that the state space is finite, but her proposition is not applicable to the present setting. Dana and Meilijson (2003, Proposition 5) proved it for the infinite state space, but the utility functions they constructed are not guaranteed to satisfy the Inada condition. To guarantee the Inada condition, it is necessary to assume that each $f_{\lambda i} : D \to D_i$ is an onto function.

$\eta_1 = \cdots = \eta_I$, and the η_i not necessarily being equal to zero. By Theorem 2, (9) implies that

$$\frac{\mu_t^i}{c_t^i} = \frac{1 - \dfrac{\underline{d}_i}{c_t^i}}{1 - \dfrac{\underline{d}}{e_t}} \frac{\mu(e_t, t)}{e_t} = \frac{1 - \dfrac{\underline{d}_i}{m_i(e_t - \underline{d}) + \underline{d}_i}}{1 - \dfrac{\underline{d}}{e_t}} \frac{\mu(e_t, t)}{e_t} = \frac{e_t - \dfrac{\underline{d}}{Im_i + (1 - Im_i)\underline{d}/e_t}}{e_t - \underline{d}} \frac{\mu(e_t, t)}{e_t}$$

for some $m_i \in (0, 1)$ with $\sum_i m_i = 1$. Here

$$\frac{\underline{d}}{Im_i + (1 - Im_i)\underline{d}/e_t}$$

is an decreasing function of e_t if $(1 - Im_i)\underline{d} > 0$, and it is an increasing function of e_t if $(1 - Im_i)\underline{d} < 0$. Thus, in particular, if the subsistence level is positive and consumer i is richer than average, that is, $\underline{d} > 0$ and $m_i > 1/I$, then the conditionally expected instantaneous consumption growth rate is an increasing function of e_t. On the other hand, if the subsistence level is still positive and yet consumer i is poorer than average, that is, $\underline{d} > 0$ and $m_i < 1/I$, then it is an increasing function of e_t.

Kurosaki (2001) assumed that every consumer exhibits constant relative risk aversion but allowed it to vary across consumers. He exploited the linear relationship, which can be derived with some manipulation, between the log of each individual consumer's consumption levels and the average of the log of all consumers' consumption levels and did not reject the hypothesis that the consumers have heterogeneous relative risk aversion. As we have pointed out earlier, however, the individual consumption growth rates, not the logs thereof, are more dispersed than are the reciprocals of relative risk aversion.

6.2 Risk-Free Interest Rates

Let $\pi = (\pi_t)_{t \in \boldsymbol{R}_{++}}$ be the state price deflator, which, after multiplying a positive constant, we can assume to be equal to $(\exp(-\rho t)u'_\lambda(e_t))_{t \in \boldsymbol{R}_{++}}$. Define $g : D \times \boldsymbol{R}_+ \to \boldsymbol{R}_{++}$ by

$$g(x, t) = \exp(-\rho t)u'_\lambda(x),$$

then $\pi_t = g(e_t, t)$. Moreover,

$$\begin{aligned}
\frac{\partial g(x, t)}{\partial x} &= \exp(-\rho t)u''_\lambda(x), \\
\frac{\partial^2 g(x, t)}{\partial x^2} &= \exp(-\rho t)u'''_\lambda(x), \\
\frac{\partial g(x, t)}{\partial t} &= -\rho \exp(-\rho t)u'_\lambda(x).
\end{aligned}$$

Thus, by Ito's Lemma, π is an Ito process with

$$
\begin{aligned}
d\pi_t &= \left(\frac{\partial g(e_t,t)}{\partial t} + \frac{\partial g(e_t,t)}{\partial x}\mu(e_t,t) + \frac{1}{2}\frac{\partial^2 g(e_t,t)}{\partial x^2}(\sigma(e_t,t))^2 \right) dt + \frac{\partial g(e_t,t)}{\partial x}\sigma(e_t,t)\, dB_t \\
&= \left(-\rho \exp(-\rho t) u'_\lambda(e_t) + \exp(-\rho t) u''_\lambda(e_t)\mu(e_t,t) + \frac{1}{2}\exp(-\rho t) u'''_\lambda(e_t)\left(\sigma(e_t,t)\right)^2 \right) dt \\
&\quad + \exp(-\rho t) u''_\lambda(e_t)\sigma(e_t,t)\, dB_t \\
&= -\pi_t \left(\left(\rho - \mu_t(e_t,t)\frac{u''_\lambda(e_t)}{u'_\lambda(e_t)} - \frac{(\sigma(e_t,t))^2}{2}\frac{u'''_\lambda(e_t)}{u'_\lambda(e_t)} \right) dt - \sigma(e_t,t)\frac{u''_\lambda(e_t)}{u'_\lambda(e_t)}\, dB_t \right).
\end{aligned}
$$

The price at time t of the discount bond maturing at time $\tau > t$ is $E_t(\pi_\tau)/\pi_t$. Hence the continuously compounded conditionally expected interest rate is $-(\tau - t)^{-1}\log\left(E_t(\pi_\tau)/\pi_t\right)$. Just as we did for (6), we can show that

$$
\lim_{\tau \downarrow t} -\frac{1}{\tau - t}\frac{E_t(\pi_\tau)}{\pi_t} = \rho - \mu_t(e_t,t)\frac{u''_\lambda(e_t)}{u'_\lambda(e_t)} - \frac{(\sigma(e_t,t))^2}{2}\frac{u'''_\lambda(e_t)}{u'_\lambda(e_t)}.
$$

Define a process $r = (r_t)_{t \in \boldsymbol{R}_+}$ by letting r_t equal to the right hand side of this equality for every $t \in \boldsymbol{R}_+$. Then r is often called the *short-rate process*. Define $h : D \times \boldsymbol{R}_+ \to \boldsymbol{R}$ by

$$
h(x,t) = \rho + \frac{\mu(x,t)}{s_\lambda(x)} - \frac{1}{2}\left(\frac{\sigma(x,t)}{s_\lambda(x)}\right)^2 \left(1 + s'_\lambda(x)\right) \tag{12}
$$

for every $(x,t) \in D \times \boldsymbol{R}_+$.

Proposition 3 *For every $t \in \boldsymbol{R}_+$,*

$$
r_t = h(e_t,t). \tag{13}
$$

Proof of Proposition 3 By definition, $u''_\lambda(e_t)/u'_\lambda(e_t) = -1/s_\lambda(e_t)$. By differentiating both sides of $u'_\lambda(x) = -u''_\lambda(x)s_\lambda(x)$ with respect to x, we obtain

$$
\frac{u'''_\lambda(e_t)}{u'_\lambda(e_t)} = \frac{1 + s'_\lambda(e_t)}{\left(s_\lambda(e_t)\right)^2}.
$$

These two equalities establish (13). ///

Proposition 3 shows that the instantaneous risk-free interest rate r_t is a function of time t and the current aggregate endowment e_t. If both μ and σ are independent of time, then so is h and hence the interest rate is a function of the current aggregate endowment alone.

In the rest of this subsection, we maintain the assumption of linear absolute risk tolerance (4) for every consumer i. The symbols, $\overline{\kappa}$, $\underline{\kappa}$, $\overline{I}$, and $\underline{I}$ are defined as in Section 5. We further assume that $\underline{\kappa} > 0$, so that $\underline{d} > -\infty$ and $\overline{d} = \infty$.

Example 1 Assume that e is the geometric Brownian motion (5) and that $\kappa_1 = \cdots = \kappa_I$, which we denote by κ. Then the mutual fund theorem holds and $s_\lambda(x) = \kappa(x - \underline{d})$ for the same

κ.[9] Moreover, h does not depend on t and we can write

$$h(x) = \rho + \frac{\hat{\mu} x}{\kappa(x - \underline{d})} - \frac{1}{2}\left(\frac{\hat{\sigma} x}{\kappa(x - \underline{d})}\right)^2 (1+\kappa) = \rho + \frac{\hat{\mu}}{\kappa(1 - \underline{d}/x)} - \frac{\hat{\sigma}^2}{2}\frac{1+\kappa}{(\kappa(1-\underline{d}/x))^2}.$$

Based on this, we can make the following observations. First, the interest rate is constantly equal to

$$\rho + \frac{\hat{\mu}}{\kappa} - \frac{\hat{\sigma}^2}{2}\frac{1+\kappa}{\kappa^2}. \tag{14}$$

if $\underline{d} = 0$. But this is the case where u_λ exhibits constant relative risk aversion $1/\kappa$. This property has been well known. Otherwise, the interest rate does indeed fluctuate as so does the aggregate endowment e_t. This is the general case of linear absolute risk tolerance, which Schmedders (2005) took up to investigate the failure of the mutual fund theorem for short-term discount bonds. Even in this case, however,

$$h(x) \to \rho + \frac{\hat{\mu}}{\kappa} - \frac{\hat{\sigma}^2}{2}\frac{1+\kappa}{\kappa^2}$$

as $x \to \infty$. Thus, in a growing economy with $e_t \to \infty$ almost surely as $t \to \infty$, the interest rate converges to the constant rate of the economy of a representative consumer exhibiting constant relative risk aversion $1/\kappa$.

In the more general case in which the mutual fund theorem fails, we obtain the following asymptotic results on the interest rates.

Proposition 4 *1. If $e_t \to \infty$ almost surely as $t \to \infty$ and if there exist a $\overline{\mu} \in \boldsymbol{R}$ and a $\overline{\sigma} \in \boldsymbol{R}$ such that $\mu(x,t)/x \to \overline{\mu}$ and $\sigma(x,t)/x \to \overline{\sigma}$ whenever $x \to \infty$ and $t \to \infty$, then*

$$r_t \to \rho + \frac{\overline{\mu}}{\overline{\kappa}} - \frac{\overline{\sigma}^2}{2}\frac{1+\overline{\kappa}}{\overline{\kappa}^2} \tag{15}$$

almost surely as $t \to \infty$.

2. If $e_t \to \underline{d}$ almost surely as $t \to \infty$ and if there exist a $\underline{\mu} \in \boldsymbol{R}$ and a $\underline{\sigma} \in \boldsymbol{R}$ such that $\mu(x,t)/(x-\underline{d}) \to \underline{\mu}$ and $\sigma(x,t)/(x-\underline{d}) \to \underline{\sigma}$ whenever $x \to \underline{d}$ and $t \to \infty$, then

$$r_t \to \rho + \frac{\underline{\mu}}{\underline{\kappa}} - \frac{\underline{\sigma}^2}{2}\frac{1+\underline{\kappa}}{\underline{\kappa}^2} \tag{16}$$

almost surely as $t \to \infty$.

The first part of this proposition states that in a growing economy in which the instantaneous conditional mean and standard deviation of the growth rate of aggregate endowments converge to $\overline{\mu}$ and $\overline{\sigma}$, the instantaneous risk-free interest rate converges to the (deterministic and constant) interest rate of the economy in which the aggregate endowment process is the geometric

[9] To be exact, we need to assume also that $\underline{d} \leq 0$, so that $e \in D$ almost surely in the sense of part 3 of Definition 1.

Brownian motion with parameters $\overline{\mu}$ and $\overline{\sigma}$ and the representative consumer exhibits constant relative risk aversion equal to the reciprocal of the highest individual absolute cautiousness in the economy. The second part states that in a contracting economy in which the instantaneous conditional mean and standard deviation of the growth rate of aggregate endowments converge to $\underline{\mu}$ and $\underline{\sigma}$, the instantaneous risk-free interest rate converges to the (deterministic and constant) interest rate in the economy in which the aggregate endowment process is the geometric Brownian motion with $\underline{\mu}$ and $\underline{\sigma}$ and the representative consumer exhibits constant relative risk aversion equal to the reciprocal of the lowest individual absolute cautiousness in the economy. In the special case where e is the geometric Brownian motion with parameters $\hat{\mu}$ and $\hat{\sigma}$ and every consumer exhibit constant relative risk aversion $1/\kappa_i$, this proposition implies that when $\hat{\mu} > \hat{\sigma}^2/2$, the instantaneous risk-free interest rate converges to what would be the (deterministic and constant) interest rate of the economy were it to consist only of the consumer having the lowest constant relative risk aversion; and that when $\hat{\mu} < \hat{\sigma}^2/2$, the instantaneous risk-free interest rate converges to what would be the (deterministic and constant) interest rate of the economy were it to consist only of the consumer having the highest constant relative risk aversion. Wang (1996), in Section 4, found this property by analytically solving for the interest rate, for the case of two consumers having constant relative risk aversion 1 and 1/2 and some prespecified utility weights λ_1 and λ_2. Although we do not have any analytical solution for interest rates, Proposition 4 shows that the same asymptotic properties for interest rates are valid for a much wider class of economies.

Proof of Proposition 4 1. By Proposition 3, it suffices to show that

$$h(x,t) \to \rho + \frac{\overline{\mu}}{\overline{\kappa}} - \frac{\overline{\sigma}^2}{2}\frac{1+\overline{\kappa}}{\overline{\kappa}^2}$$

whenever $x \to \infty$ and $t \to \infty$. Indeed, by part 1 of Proposition 2, $s'_\lambda(x) \to \overline{\kappa}$. Since $\overline{\kappa} > 0$, this implies that $s_\lambda(x) \to \infty$. Hence, by L'Hôpital's rule, $s_\lambda(x)/x \to \overline{\kappa}$. Therefore,

$$h(x,t) = \rho + \frac{\mu(x,t)/e_t}{s_\lambda(x)/x} - \frac{1}{2}\left(\frac{\sigma(x,t)/x}{s_\lambda(x)/x}\right)^2 \left(1 + s'_\lambda(x)\right) \to \rho + \frac{\overline{\mu}}{\overline{\kappa}} - \frac{\overline{\sigma}^2}{2}\frac{1+\overline{\kappa}}{\overline{\kappa}^2}.$$

2. By Lemma 2 of HHK, which is due to Wilson (1968), $s_\lambda(x) = \sum s_i(f_{\lambda i}(x))$. Since $f_{\lambda i}(x) \to \underline{d}_i$ as $x \to \underline{d}$, $s_i(f_{\lambda i}(x)) \to 0$ as $x \to \underline{d}$. Thus $s_\lambda(x) \to 0$ as $x \to \underline{d}$. By L'Hôpital's rule, $s_\lambda(x)/(x - \underline{d}) \to \underline{\kappa}$ as $x \to \underline{d}$. Hence

$$h(x,t) = \rho + \frac{\mu(x,t)/(x-\underline{d})}{s_\lambda(x)/(x-\underline{d})} - \frac{1}{2}\left(\frac{\sigma(x,t)/(x-\underline{d})}{s_\lambda(x)/(x-\underline{d})}\right)^2 \left(1 + s'_\lambda(x)\right) \to \rho + \frac{\underline{\mu}}{\underline{\kappa}} - \frac{\underline{\sigma}^2}{2}\frac{1+\underline{\kappa}}{\underline{\kappa}^2}.$$

///

Having identified the asymptotic behavior of the instantaneous risk-free interest rate processes, we now ask whether we can correctly predict the instantaneous risk-free interest rates by postulating that the representative consumer's utility function u_λ exhibit hyperbolic absolute risk aversion or constant relative risk aversion. The importance of this question arises from the prevalent use of such classes of utility functions in the general equilibrium model of asset pricing, presumably for their analytical tractability. In particular, if we take the representative consumer as a primitive concept of the model and impose some tractable functional form on his utility function (as done by Metra and Prescott (1985), for example), rather than employing a closer-to-reality approach of constructing it from the explicitly modelled group of heterogeneous consumers, then we should be well aware of any potential biases in our estimation of interest rates. In the rest of this subsection, we shall clarify the nature of such biases in a somewhat informal manner.

First, we can immediately see from (12) that for any given level x of the current aggregate endowment e_t, if we know the representative consumer's absolute risk tolerance $s_\lambda(x)$ and absolute cautiousness $s'_\lambda(x)$, then we can correctly derive the instantaneous risk-free interest rate via $r_t = h(e_t, t)$. In other words, the assumption of linear absolute risk tolerance allows us to correctly predict the interest rate at a given level of current aggregate endowments if the representative consumer's absolute risk tolerance and the absolute cautiousness at the level are known. We argue, however, that the assumption of constant relative risk aversion (or, equivalently, linear absolute risk tolerance with the constant term being zero) leads us to overestimate the interest rate at a given level of current aggregate endowments, even if the representative consumer's relative risk tolerance at the level is known.

Using relative risk tolerance, it is then to derive from (12) that $h(x,t)$ is equal to

$$\rho + \frac{\mu(x,t)}{x}\frac{1}{q_\lambda(x)} - \frac{1}{2}\left(\frac{\sigma(x,t)}{x}\right)^2 \frac{1}{q_\lambda(x)}\left(1 + \frac{1}{q_\lambda(x)}\right) - \frac{1}{2}\left(\frac{\sigma(x,t)}{x}\right)^2 \frac{1}{q_\lambda(x)}\frac{q'_\lambda(x)x}{q_\lambda(x)}. \tag{17}$$

The first two terms involves $q_\lambda(x)$ but not $q'_\lambda(x)$. Thus, even if we assume that the representative consumer exhibits constant relative risk aversion (or, equivalently, constant relative risk tolerance), we can correctly predict the values of these two terms as long as the relative risk aversion at the level is known. But the last term involves the elasticity of the relative risk tolerance, and thus $q'_\lambda(x)$. Proposition 7 of HHK (which is derived from (6.10) of Calvet, Grandmont, and Lemaire (1999)) implies that $q'_\lambda(x) \geq 0$ if all the $q'_i(f_{\lambda i}(x))$ are nonnegative (including the case of constant relative risk aversion); and $q'_\lambda(x) > 0$ unless they are completely equal. Therefore, if we assume that the representative consumer exhibits constant relative risk aversion, then we miss out this term, overestimating the interest rate.

Remark 4 Weil (1992) also showed that the representative consumer model may overestimate the risk-free interest rate. The reason for this overestimation is, however, different from ours. He considered an economy with a continuum of consumers who are ex ante homogeneous but ex post heterogeneous, in the sense that the shocks to their endowments, interpreted as labor incomes, are independently and identically distributed. The asset markets are incomplete and these idiosyncratic risks cannot be insured. The exposure to these risks increases the consumers' precautionary saving demand for the risk-free bond, leading to a lower equilibrium interest rate than is predicted by the representative-consumer model in which the idiosyncratic risks can be pooled together. The crucial property for utility functions to increase the precautionary saving demand is prudence, but the crucial property in the above argument was constant relative risk aversion.

Remark 5 As can be seen from (14) (and also from (17)), if an economy consists of only one consumer exhibiting constant relative risk aversion $1/\kappa$ and the endowment process is a geometric Brownian motion, then the instantaneous risk-free interest rate process is constant (and deterministic), and its level is a quadratic function of the constant relative risk aversion $1/\kappa$. If, moreover, the economy is growing so that the parameters $\hat{\mu}$ and $\hat{\sigma}$ satisfy $\hat{\mu} > \hat{\sigma}^2/2$, then the quadratic function attains the maximum at $1/\kappa = \hat{\mu}/\hat{\sigma}^2 - 1/2 > 0$. Thus, if we take $\hat{\mu}$ and $\hat{\sigma}$ so that $\hat{\mu}/\hat{\sigma}^2 - 1/2 = 3/4$, then both $1/\kappa_1 = 1$ and $1/\kappa_2 = 1/2$ give rise to the same constant interest rate. Let's denote this level by r^*. Now, one of the examples in Section 4 of Wang (1996) shows that in an economy consisting of these two consumers, the interest rate may always fluctuate at levels lower than r^*. This intriguing phenomenon is a consequence of heterogeneous risk attitudes, but its extent is more than what can be explained by the observation in the paragraph of (17), for the following reason: Since $1/\kappa_1 = 1$, $1/\kappa_2 = 1/2$, and $1/q_\lambda(x)$ is an weighted average of these two,[10] $1/q_\lambda(x)$ lies strictly between 1/2 and 1. In the economy in which the representative consumer has constant relative risk aversion $1/q_\lambda(x)$, the constant interest rate, which we denote by r^x, is higher than r^*, because, as seen before, it is a quadratic function of constant relative risk aversion attaining the maximum between 1/2 and 1. The observation in the paragraph of (17) implies that the interest rate in the two-consumer economy is lower than r^x, but it falls short of implying that the interest rate is lower than r^*. We can therefore conclude that the extent of reduction in interest rates caused by heterogeneous risk attitudes may well be larger than the reduction explained by the observation in the paragraph of (17).

Our finding so far can be summarised as follows: We can correctly predict the instantaneous risk-free interest rate by approximating the representative consumer's absolute risk tolerance by a linear function with a possibly nonzero constant term, but we cannot do so when constrained to use a linear function with a zero constant term. This result provides a case for the use of utility functions exhibiting hyperbolic absolute risk aversion, which constitute a two-parameter family of utility functions, rather than those exhibiting constant relative risk aversion, which constitute a one-parameter sub-family, to approximate the representative consumer's risk attitudes.

We now proceed to show that when it comes to analyzing how the interest rate responds to the fluctuations in aggregate endowments, even the two-parameter family of hyperbolic risk

[10]This can be easily derived from the results in HHK.

aversion is not sufficient. For this purpose, let's look into the partial derivative $\partial h(x,t)/\partial x$:

$$
\begin{aligned}
&\frac{\partial h}{\partial x}(x,t)\\
&=\frac{\frac{\partial \mu}{\partial x}(x,t)}{s_\lambda(x)}-\mu(x,t)\frac{s'_\lambda(x)}{(s_\lambda(x))^2}-\sigma(x,t)\frac{\partial \sigma}{\partial x}(x,t)\frac{1+s'_\lambda(x)}{(s_\lambda(x))^2}\\
&\quad-\frac{(\sigma(x,t))^2}{2}\left(\frac{s''_\lambda(x)}{(s_\lambda(x))^2}-(1+s'_\lambda(x))\frac{2s'_\lambda(x)}{(s_\lambda(x))^3}\right)\\
&=\frac{\mu(x,t)}{s_\lambda(x)}\left(\frac{\frac{\partial \mu}{\partial x}(x,t)}{\mu(x,t)}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)-(1+s'_\lambda(x))\left(\frac{\sigma(x,t)}{s_\lambda(x)}\right)^2\left(\frac{\frac{\partial \sigma}{\partial x}(x,t)}{\sigma(x,t)}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)\\
&\quad-\frac{1}{2}\left(\frac{\sigma(x,t)}{s_\lambda(x)}\right)^2 s''_\lambda(x). \qquad (18)
\end{aligned}
$$

Here the last term captures the extra responsiveness of the interest rate due to the convex absolute risk tolerance. Theorem 5 of HHK implies that $s''_\lambda(x) \geq 0$ by the assumption of hyperbolic absolute risk aversion (4), and also that $s''_\lambda(x) > 0$ unless all the κ_i are identical. Therefore, if we assume that the representative consumer exhibits hyperbolic absolute risk aversion, then we miss out this term, overestimating the change in the interest rate caused by the change in aggregate endowments. Although this result does not always imply that the absolute value $|\partial h(x,t)/\partial x|$ is underestimated by assuming hyperbolic absolute risk aversion, we still obtain the following result.

Claim If (4) is satisfied for every i, all the κ_i are not completely equal, $\underline{d}=0$, e is the geometric Brownian motion (5), and

$$\hat{\mu} \geq \frac{1+\overline{\kappa}}{\underline{\kappa}}\hat{\sigma}^2, \qquad (19)$$

then

$$\left|\frac{\partial h}{\partial x}(x,t)\right| > \left|\frac{\mu(x,t)}{s_\lambda(x)}\left(\frac{\frac{\partial \mu}{\partial x}(x,t)}{\mu(x,t)}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)-(1+s'_\lambda(x))\left(\frac{\sigma(x,t)}{s_\lambda(x)}\right)^2\left(\frac{\frac{\partial \sigma}{\partial x}(x,t)}{\sigma(x,t)}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)\right|$$

for every $(x,t) \in \boldsymbol{R}_{++} \times \boldsymbol{R}_{+}$.

This claim means that ignoring the third term always underestimates $|\partial h(x,t)/\partial x|$. Note that $\underline{d}=0$ if, but not only if, all consumers exhibit constant relative risk aversion.

To prove this claim, it suffices to show that

$$\frac{\mu(x,t)}{s_\lambda(x)}\left(\frac{\frac{\partial \mu}{\partial x}(x,t)}{\mu(x,t)}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)-(1+s'_\lambda(x))\left(\frac{\sigma(x,t)}{s_\lambda(x)}\right)^2\left(\frac{\frac{\partial \sigma}{\partial x}(x,t)}{\sigma(x,t)}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right) \leq 0. \qquad (20)$$

Indeed, the left hand side of (20) is equal to

$$
\begin{aligned}
&\frac{\hat{\mu}x}{s_\lambda(x)}\left(\frac{\hat{\mu}}{\hat{\mu}x}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)-(1+s'_\lambda(x))\left(\frac{\hat{\sigma}x}{s_\lambda(x)}\right)^2\left(\frac{\hat{\sigma}}{\hat{\sigma}x}-\frac{s'_\lambda(x)}{s_\lambda(x)}\right)\\
=&\frac{\hat{\mu}}{s_\lambda(x)}\left(1-\frac{s'_\lambda(x)x}{s_\lambda(x)}\right)-(1+s'_\lambda(x))\left(\frac{\hat{\sigma}}{s_\lambda(x)}\right)^2 x\left(1-\frac{s'_\lambda(x)x}{s_\lambda(x)}\right)\\
=&-\frac{1}{s_\lambda(x)}\left(\hat{\mu}-\hat{\sigma}^2\frac{1+s'_\lambda(x)}{s_\lambda(x)/x}\right)\left(\frac{s'_\lambda(x)x}{s_\lambda(x)}-1\right).
\end{aligned}
$$

As mentioned above, s_λ is a strictly convex function. As shown in the proof of part 2 of Proposition 4, $s_\lambda(x) \to 0$ as $x \to 0$. Hence the elasticity of s_λ is everywhere greater than 1, implying that

$$\frac{s'_\lambda(x)x}{s_\lambda(x)}-1>0.$$

Hence (20) holds whenever

$$\hat{\mu}-\hat{\sigma}^2\frac{1+s'_\lambda(x)}{s_\lambda(x)/x}>0. \tag{21}$$

By (10),

$$\frac{1+s'_\lambda(x)}{s_\lambda(x)/x}<\frac{1+\overline{\kappa}}{\underline{\kappa}}.$$

Now (21) follows from this and (19).

The importance of correctly predicting $|\partial h(x,t)/\partial x|$ can be seen in terms of the volatility of the interest rate process r. Indeed, by Proposition 3, r is an Ito process with

$$dr_t=\left(\frac{\partial h}{\partial t}(e_t,t)+\frac{\partial h}{\partial x}(e_t,t)\mu(e_t,t)+\frac{(\sigma(e_t,t))^2}{2}\frac{\partial^2 h}{\partial x^2}(e_t,t)\right)dt+\frac{\partial h}{\partial x}(e_t,t)\sigma(e_t,t)\,dB_t.$$

Thus the conditional instantaneous variance of r_t is equal to

$$\left(\frac{\partial h}{\partial x}(e_t,t)\right)^2(\sigma(e_t,t))^2.$$

Ignoring the convexity of the representative consumer's absolute risk tolerance may lead us to underestimate the volatility of the interest rate process.

7 Conclusion

We have shown (Theorem 1) that every continuous-time model in which all consumers have time-separable and time-homogeneous expected utility functions with a common probabilistic belief and a common discount rate can be reduced to a static model in which they have expected utility functions with a homogeneous probabilistic belief. This result allowed us to derive some properties (Propositions 1 and 2) on the efficient-risk sharing rules and the representative consumer's risk attitudes in the continuous-time model from the corresponding results in the static model. In the special case where the aggregate endowment is an Ito process, we also investigated (in Section 6) the implications on the level and volatility of interest rates and the conditionally expected instantaneous consumption growth rates.

There are, of course, some issues that have not been addressed to in this paper. We now list three such issues, thereby suggesting possible directions of future research.

First, although we explored how the individual consumptions evolve over time, we did not explicitly model financial markets and thus did not identify the dynamic asset trading strategies that implement those individual consumption processes. Identifying such trading strategies would attach more practical relevance to our results.

Second, we are did not analyze the evolution of individual wealth processes. A consumer's wealth is defined as the total market value of his current portfolio, which is, under the budget constraint admitting no arbitrage opportunity, equal to the discounted value of all consumptions subtracted by, if any, other sources of income, such as labor income. We saw in Proposition 2 that in a continuous-time economy in which the consumers have differing levels of constant absolute cautiousness and the aggregate endowment diverges to infinity almost surely, the consumption share is eventually concentrated on those consumers with the highest absolute cautiousness. Once appropriate assumptions are made on initial endowments (of both financial assets and other sources of income), it will be worthwhile, as in Sections 4 and 5 of Dumas (1989), to attempt to explore implications of this fact onto the evolution of wealth processes.

Finally, a thorough analysis of the term structure of interest rates should be conducted. Although we obtained the current instantaneously risk-free interest rate as a function of time and the current aggregate endowments, we did not explore the zero-coupon bond prices with maturities at a distant future. Also, in the continuous-time setting, there is a vast literature of interest rate models, in which various types of short-rate processes are postulated without any consideration on utility maximization, let alone heterogeneous risk aversion. The relationship between the representative consumer's risk aversion and the existing continuous-time term structure models, especially those assuming Markovian properties, should be clarified.

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