A Dynamic General Equilibrium Model with Centralized Auction Markets*

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Abstract

A model of centralized auction markets with a large number of agents is considered. In the model, there is no friction in trades in the sense that there is no transaction cost and no asymmetry of information. As contrasted with a conventional wisdom, which says that the outcome is Walrasian regardless of institutional differences as far as competitive conditions are met, the equilibrium in our model does not coincide with that in the Walrasian market. More precisely, the set of equilibrium in our model with auction markets is a continuum which includes the Walrasian equilibria. A model of decentralized auction markets with a large number of agents is also investigated and a similar result is obtained.

Keywords: Dynamic General Equilibrium, Auction, Walrasian Market, Real Indeterminacy of Stationary Equilibria

Journal of Economic Literature Classification Number: C72, C78, D44, D51, D83, E40

1 Introduction

In the literature of decentralized trading, the equilibrium price is considered as an equilibrium outcome of some noncooperative bargaining game and it is investigated whether the outcome coincides with that in the centralized Walrasian market. (See, for example, Rubinstein and Wolinsky [18] [19] and Gale [4] [5] [6].) Before Rubinstein and Wolinsky [18], economic theorists had intuitively believed that the outcome is Walrasian regardless of institutional differences as far as competitive conditions are met. Here the competitive conditions mean that there is a large number of agents and that markets are frictionless, e.g., there is no transaction cost and no asymmetry of information.

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Rubinstein and Wolinsky refute this claim by showing that the outcome is not Walrasian in a random matching model, where matched agents bargain over the terms of trade. Gale [4] [5] [6] presents other frictionless and decentralized market models in which the equilibrium outcome is Walrasian. On the other hand, as for centralized markets, we believe that it is still a conventional wisdom that any centralized market outcome is Walrasian regardless of institutional differences as far as the competitive conditions are met. In this paper, we refute this claim by presenting a model of centralized auction markets with a large number of agents. In the model, there is no friction in trades in the sense that there is no transaction cost and no asymmetry of information. As contrasted with the conventional wisdom, the equilibrium in the model does not coincide with that in the Walrasian market. A model of decentralized auction markets with a large number of agents is also investigated and a similar result is obtained.

Recently, real indeterminacy of stationary equilibria has been found in both specific and general search models with divisible money. (See, for example, Green and Zhou [8] [9], Kamiya and Shimizu [13], Matsui and Shimizu [15], and Zhou [23].) In other words, if we assume the divisibility of money in these models, the stationary equilibria become indeterminate. However, in models with centralized markets and with divisible money, indeterminacy of stationary equilibria has not been found.

To the best of our knowledge, Edgeworth [3] first presents an institutional environment different from the Walrasian market, where the outcome converges to the Walrasian equilibrium as the number of agents becomes large. More precisely, he shows that the set of core allocations converges to that of Walrasian equilibrium in a simple environment. Then Shubik [21], Debreu and Scarf [2], and Hildenbrand [11] shed a light on the Edgeworth's classical work from the view point of modern economics and show that the Edgeworth's result holds in considerably general frameworks. Note that Aumann [1] directly assumes that there is a continuum of agents and shows that the set of core allocations exactly coincides with that of Walrasian equilibrium. There are some other institutional environments, where the outcome converges to the Walrasian equilibrium: Novshek and Sonnenschein [16] and Shapley and Shubik [20] investigate a Cournot competition model with a large number of firms and a trading post model with a large number of agents, respectively. Thus we have several models which confirm the conventional wisdom. However, it has never been proved that the conventional wisdom holds true in sufficiently general frameworks. In this paper, by presenting a counter example, we show that the conventional wisdom does not necessarily hold.

Our model is a dynamic general equilibrium model with fiat money and a finite number of perishable goods. For each good, there is a centralized market, where goods are traded by the uniform price auction using fiat money. Each agent can produce a type of good which she cannot consume and can consume another type of good. In each period, she can visit just one centralized market. For example, at period t, if an agent do not have money, she would first visit the market of her production good and then in period $t+1$ she would buy her consumption good using money she obtained at t. [1] The conditions for a stationary equilibrium are (i) each agent maximizes the expected value of utility-streams, (ii) the money holdings distribution of the economy is stationary, i.e., time-invariant, and (iii) the total amount of money the agents have is equal to exogenously given amount of money. We compare the equilibrium with that in the Walrasian market with cash-in-advance constraint, i.e., the case that equilibria are determined at the price that demand is equal to supply and the expenditure of each consumer is constrained by the amount of her money holding. We show that the set of equilibria with auction markets is a continuum while that of the Walrasian market model is a singleton included in the set of equilibria with the auction markets, i.e., the sets of equilibria do not coincide.

Marshall [14] defines the tripartite division of time: a day, a short period, and a long period. (See also Hicks [10].) In a day the production decision cannot be changed, in a short period the amount of production good can be changed for a given capital, and in a long period the amount of capital can also be changed. Rubinstein-Wolinsky-Gale models are intrinsically static models; for a given endowment, each agent seeks for a better consumption bundle by trading with randomly matched agents. (See Gale [7].) Thus time in their models is in a sense within a day, say hours. As contrasted with their models, our model is intrinsically a dynamic model; for each time period an agent can produces a good and can trade in a centralized market. Thus the time is in a short period, i.e., time between days. (Note that in our model there is no distinction between a short period and a long period, since there is no capital in our model.) Note that, in Rubinstein-Wolinsky-Gale models, a discount factor less than one implies that there is trade friction, since it is for within a day and consumption might occur late in the day. On the other hand, in our model a discount factor less than one does not imply trade friction because it is for between days. In other words, in our model a discount factor within a day is one and thus there is no trade friction.

[1] Even without this limited participation constraint, we can obtain almost the results. See Section B in Appendix.

Gale [7] considers convincing models of general equilibrium as the ones in which markets of goods are distinct, and agents do not trade most goods and do not therefore participate in most markets. In our model, there are $k \geq 3$ types of agents and the same number of types of goods; a type i agent produces type $i+1$ good and she consumes type i good. Moreover, each agent can participate just one market in each time period.

In our model, goods are indivisible and rationing might occur in the auction markets. However, our results on indeterminacy do not at all depend on indivisibility of goods or rationing but on a nature of monetary trades and divisibility of money.

The plan of this paper is as follows. In Section 2, we present a common environment. In Section 3, we investigate a dynamic general equilibrium model with auction markets and show that there is a continuum of stationary equilibria. Then in Section 4, we show that if the trading institution is the Walrasian market with cash-in-advance constraints, the stationary equilibrium is essentially unique, where the environment besides the the trading institution is the same as in Section 2. In Section 5, we investigate the logic behind the indeterminacy by classifying monetary trades into two types. Finally in Appendix we present some related models including a decentralized auction market model.

2 Environment

We present an environment common in this paper. Time is discrete denoted by $t = 1, 2, \ldots$. There is a continuum of agents of which measure is one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Only one unit of indivisible and perishable good i can be produced by a type $i-1 \pmod{k}$ agent with production cost $c > 0$. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. Let $\theta = u/c$ and assume $\theta > 1$. There is completely divisible and durable fiat money of which nominal stock is $M > 0$. Let $\gamma > 0$ be the discount factor.

3 Centralized Auction Markets

In each time period, a centralized market is open for each good. At the jth market, good j is traded by the auction we will specify below. In each period, each agent can join just one market. If a type i agent chooses to join the $(i+1)$th market, then she becomes a seller of her product, i.e., good $i+1$. If a type i agent chooses to join the ith market, then she becomes a buyer and bids a price. Each agent can also choose not to join any market. Note that an agent cannot buy and sell in the same period. However, even without this limited participation constraint, we can obtain almost the same results. See Section B in Appendix.

We consider the uniform price sealed bid auction. Since we restrict our attention to the equilibrium in which any bid on the equilibrium path is made by agents with a positive measure, we define the rule of the uniform price auction only in the following two cases:

(i) each realized bid is made by a positive measure of agents, and

(ii) only one bid is made by just one agent and the other bids are made by a positive measure of agents.

As far as we investigate a discrete money holdings distribution and a Markov strategy in which any bid is made by agents with a positive measure, it can be analyzed without the specifications in the other cases. In other words, a strategy is an equilibrium no matter what the specifications in the other cases are.

Let S be the measure of sellers, $b_1, b_2, \dots, b_L$ be bids made by positive measures of agents, and $B_1, B_2, \dots, B_L$ be the corresponding measures. Without loss of generality, we can assume $b_1 > b_2 > \cdots > b_L \geq 0$. Denote $\tilde{B}_\ell = \sum_{i=1}^{\ell} B_i$ and let $\tilde{B}_0 = 0$.

First, we consider the case (i). If $S < \tilde{B}_L$, there exists ℓ such that $S \geq \tilde{B}_{\ell-1}$ and $S < \tilde{B}_\ell$. Then the buyers in B_i, $i < \ell$, obtain the good with probability one. The buyers in B_ℓ obtain the good with probability $\frac{S-\tilde{B}_{\ell-1}}{B_\ell}$. Of course, any seller can sell her good. The uniform price is b_ℓ. If $S \geq \tilde{B}_L$, then all buyers obtains one unit of goods with price b_L. Any seller can sell her good with probability $\frac{S}{\tilde{B}_L}$.

Next, we consider the case (ii). Let $\hat{b}$ be the bid by a single agent, say Buyer 0. Clearly, the distribution of bids is the same as in the case of (i). Thus the uniform price and the buyers' possibility of winning, besides Buyer 0, is defined as in the case of (i). As for Buyer 0, if $S < \tilde{B}_L$ and $\hat{b} > b_\ell$, she can obtain the good with probability one. If $S < \tilde{B}_L$ and $\hat{b} < b_\ell$, she cannot obtain the good. Finally, if $S \geq \tilde{B}_L$, she can obtain the good with probability one.

We focus on Markov strategies, i.e., we seek for an equilibrium strategy depending only on money holdings. We also focus on stationary equilibria in which all agents with identical characteristics act similar and in which all of the k types are symmetric. Thus a candidate for an equilibrium strategy can be defined as a function of a money holding $\eta \in R_+$,

$$\xi : R_+ \to \{\sigma\} \cup (\{\beta\} \times R_+) \cup \{\nu\},$$

where $\xi(\eta) = \sigma$ implies that the agent chooses to be a seller, and $\xi(\eta) = (\beta, b)$ implies that the agent chooses to be a buyer and her bid price is b. Thus if there is a finite number of bid prices $(b_1, b_2, \ldots, b_L)$ such that $\sum_{b_i=\xi(np)} h_n > 0$, $i = 1, \ldots L$, and $B+S \leq 1$, where $B = \sum_{i=1}^{L} \sum_{(\beta,b_i)=\xi(np)} h_n$ and $S = \sum_{\sigma=\xi(np)} h_n$, then the price of each good is determined by the above rule of the uniform price auction. The price is expressed as a function of (h, ξ), denoted by $\pi(h, \xi)$. Moreover, as for sellers, the probability of selling good is expressed as a function of (h, ξ), denoted by $\nu_S(h, \xi)$. Similarly, for each bid $b \in R_+$, the probability of obtaining good is expressed as a function of (b, h, ξ), denoted by $\nu_B(b, h, \xi)$.

In this paper, we focus on stationary equilibria in which money holdings distribution has a support $\{0, p, 2p, \ldots, Np\}$, where $p > 0$ is an equilibrium price. Thus the money holdings distribution is expressed as $(h_0, h_1, \ldots, h_N)$, where h_n is the measure of agents with money holding np. The stationary equilibrium is defined as follows.

Definition 1 $\langle h, \xi, V \rangle$, where $V : R_+ \to R$ is a value function, is said to be a *stationary equilibrium with a discrete money holdings distribution* if

- h is stationary under the strategy ξ,
- $p = \pi(h, \xi)$,
- $\sum_{n=0}^{N} pnh_n = M$,
- there is a finite number of bid prices $(b_1, b_2, \ldots, b_L)$ such that $\sum_{(\beta,b_i)=\xi(np)} h_n > 0$, $i = 1, \ldots L$, and $B+S \leq 1$, where $B = \sum_{i=1}^{L} \sum_{(\beta,b_i)=\xi(np)} h_n$ and $S = \sum_{\sigma=\xi(np)} h_n$, and
- given h and ξ, the value function V, together with ξ, solves the Bellman equation, i.e., for $\eta \in R_+$,

$$\begin{aligned} V(\eta) = \max \big\{ & \nu_S(h,\xi)\,(-c + \gamma V(\eta + \pi(h,\xi))) + (1 - \nu_S(h,\xi))\gamma V(\eta), \\ & \max_{b \in R_+} \nu_B(b,h,\xi)\,(u + \gamma V(\eta - \pi(h,\xi))) + (1 - \nu_B(b,h,\xi))\gamma V(\eta), \gamma V(\eta) \big\}. \end{aligned}$$

In this section, we focus on stationary equilibria with $N = 1$. We consider the following strategy as a candidate for equilibrium:

- an agent with $\eta \in [0, p)$ chooses to be a seller, and
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer and bids η.

We consider the case of $h_0 \leq \frac{1}{2}$. This implies that the measure of buyers is larger than or equal to that of sellers, and therefore an agent with $\eta \in (0, p)$ could not win even if she had chosen to be a buyer.

The stationarity of money holdings distributions is expressed as follows. Since $\nu_B(h, \xi) = r = \frac{h_0}{1-h_0}$, the measure of agents who can buy is $(1-h_0)\frac{h_0}{1-h_0}$ and their money holdings become 0. On the other hand, since agents without money can always sell, the measure of agents who can sell is h_0 and their money holdings become p. Thus the stationarity of money holdings at 0 is

$$(1-h_0)\frac{h_0}{1-h_0} = h_0.$$

Similarly, the stationarity of money holdings at p is

$$h_0 = (1-h_0)\frac{h_0}{1-h_0}.$$

Both of them are (the same) identities and thus any (h_0, h_1) satisfying $h_0 + h_1 = 1, h_0 \geq 0$, and $h_1 \geq 0$, is a stationary distribution.

Under the strategy, the value function is expressed as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta < p, \\ r\left(u + \gamma V(0)\right) + (1-r)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p, \end{cases}$$

where

$$r = \frac{h_0}{1-h_0}.$$

Let n be the integer part of η and ι be the residual, then η is uniquely expressed as $\eta = np + \iota$. Thus the value function becomes as follows:

$$V(np+\iota) = \begin{cases} \frac{r\gamma u - (1-\gamma+r\gamma)c}{(1-\gamma)(1+r\gamma)}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{1+r\gamma}\left[(1-r+r\gamma)u + r\gamma c\right]\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^n}{1+\gamma}(u+c)\right\}, & \text{if } \iota \neq 0. \end{cases} \quad (1)$$

We need to check the following incentive conditions:

(i) incentive for an agent with $\eta \in [0, p)$ to be a seller instead of doing nothing,

(ii) incentive for an agent with $\eta \in [0, p)$ to be a seller instead of being a buyer,

(iii) incentive for an agent with $\eta \in [p, \infty)$ to be a buyer instead of doing nothing,

(iv) incentive for an agent with $\eta \in [p, \infty)$ to be a buyer instead of being a seller, and

(v) incentive for a buyer with $\eta \in [p, \infty)$ to bids η instead of bidding the other prices.

Among them, it is easily verified that (ii), (iii), (v) are satisfied. (i) and (iv) are equivalent to the following inequalities:

(I) $V(0) \geq 0$.

(II) $V(p) \geq -c + \gamma V(2p)$.

As for (I), it suffices to show that the following inequality holds:

$$\theta \geq \frac{1 - \gamma + r\gamma}{r\gamma}. \tag{2}$$

As for (II), it suffices to show that the following inequality holds:

$$(r - \gamma + r\gamma(1-\gamma))u + (1 - r\gamma^2)c \geq 0. \tag{3}$$

This is satisfied if

$$\gamma \geq \frac{1}{\theta}, \tag{4}$$

and

$$r \geq \frac{\gamma\theta - 1}{(1 + \gamma - \gamma^2)\theta - \gamma^2}. \tag{5}$$

are satisfied. Note that (4) is a sufficient condition that the coefficient of r in (3) is positive. Thus if (4) is satisfied, then (3) is equivalent to (5).

Then (2), (4), and (5) are sufficient for the incentive. Then we should find a sufficient condition on γ and θ for these inequalities.

Theorem 1 For any $\gamma \in (\frac{1}{\theta}, 1)$, there exists a stationary equilibrium with a discrete money holdings distribution with $N = 1$ for any $h_0 \in \left[\frac{\gamma\theta - 1}{(1+2\gamma-\gamma^2)\theta - 1 - \gamma^2}, \frac{1}{2}\right]$.

Proof:
Substituting (5) with equality into (2), we obtain

$$-(\theta+1)\gamma^3 + (\theta^2+\theta+1)\gamma^2 - (\theta-1)\gamma - \theta \geq 0.$$

Since the LHS of the above is equal to $((1+\theta)\gamma^2 - \gamma - 1)(\theta - \gamma)$, the above inequality holds if and only if $\gamma \leq \theta$. $\gamma \in (\frac{1}{\theta}, 1)$ clearly satisfies the above inequality. ∎

Remark 1 Marshall [14] defines the tripartite division of time: a day, a short period, and a long period. In a day the production decision cannot be changed, in a short period the amount of production good can be changed for a given capital, and in a long period the amount of capital can also be changed. As we discussed in Introduction, a discount factor less than one does not imply trade friction in our model, since it is for between days. In other words, in our model, a discount factor within a day is one and thus there is no trade friction. On the other hand, Rubinstein-Wolinsky-Gale models are intrinsically static models; for a given endowment, each agent seeks for a better consumption bundle by trading with randomly matched agents. (See Gale [7].) Thus time in their models is in a sense within a day, say hours. Thus in Rubinstein-Wolinsky-Gale models, a discount factor less than one implies that there is trade friction, since it is for within a day and consumption might occur late in the day.

4 Walrasian Markets

In the same environment as in Section 2, we define the concept of stationary Walrasian equilibrium, where a competitive market is open for each good. More precisely, (a) each agent maximizes the discounted sum of utility stream for given prices of goods under the budget constraint and the cash-in-advance constraint, (b) the markets of goods clear, i.e., for each good, the measure of sellers is equal to that of buyers, (c) the money demand is equal to supply, and (d) the money holdings distribution and the price of good are stationary, i.e., time-invariant. Moreover, as in Section 3, we assume each agent can only join just one market in each period.

As in the previous section, we focus on stationary equilibria in which all agents with identical characteristics act similar and in which all of the k types are symmetric. Thus we seek for equilibria, where the prices of goods are the same. Let the price be $p \in R_+$. For a given p, the behavior of an agent with $\eta \in R_+$ is expressed in terms of a Bellman equation as follows:

$$V(\eta) = \max_{(\chi,\zeta)\in C} \chi u - \zeta c + \gamma V(\eta'), \tag{6}$$
$$\text{s.t. } \chi p + \eta' = \eta + \zeta p,\ \chi p \leq \eta,\ \eta' \geq 0,$$

where $(\chi, \zeta) = (1, 0)$ when an agent is a buyer, $(\chi, \zeta) = (0, 1)$ when she is a seller, and $(\chi, \zeta) = (0, 0)$ when she does nothing. Since she cannot choose both a buyer and a seller, $C = \{(1, 0), (0, 1), (0, 0)\}$. The above Bellman equation is expressed by

$$V(\eta) = \begin{cases} \max\{u + \gamma V(\eta - p), -c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta - p \geq 0, \\ \max\{-c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta - p < 0. \end{cases}$$

For a given p, the unique value function $V : R_+ \to R$ and the optimal policy correspondence $\phi : R_+ \to \{\beta, \sigma, \nu\}$ are obtained, where β, σ, and ν stand for 'buy', 'sell', and 'do nothing', respectively. Let $\mathcal{B}$ be the Borel σ-algebra on $[0, \infty)$. A transition function $T : R_+ \times \mathcal{B} \to [0, 1]$[2] is said to be consistent with ϕ if, for any $A \in \mathcal{B}$, $T(\eta, A)$ is positive only if

- $\sigma \in \phi(\eta)$ and $\eta + p \in A$, or
- $\beta \in \phi(\eta)$ and $\eta - p \in A$, or
- $\nu \in \phi(\eta)$ and $\eta \in A$.

Stationary Walrasian equilibrium is defined as follows.

Definition 2 $\langle p, F, V, \phi, T\rangle$, where F is a probability measure on $\mathcal{B}$, is said to be a *stationary Walrasian equilibrium* if

(i) $\phi : R_+ \to \{\beta, \sigma, \nu\}$ is the optimal policy correspondence associated with V,

(ii) $T : R_+ \times \mathcal{B} \to [0, 1]$ is consistent with ϕ,

(iii) F is stationary under T,

(iv) $\int_{n=0}^{\infty} \eta dF = M$,

(v) given p, the value function V solves the Bellman equation,

[2] A transition function $T : R_+ \times \mathcal{B} \to [0, 1]$ is a function such that

- for each $\eta \in R_+$, $T(\eta, \cdot)$ is a probability measure on $(R_+, \mathcal{B})$, and
- for each $A \in \mathcal{B}$, $T(\cdot, A)$ is a $\mathcal{B}$-measurable function.

For the details, see Stokey and Lucas [22].

(vi) $T(\eta, \{\eta - p\})$ and $T(\eta, \{\eta + p\})$ are measurable functions of η, and

$$\int T(\eta, \{\eta - p\})dF = \int T(\eta, \{\eta + p\})dF$$

holds.

As for (i)-(v), no explanation is needed. (vi) is the market clearing condition for goods; namely, the measure of buyers is equal to that of sellers. Note that, by Walras law, money demand is equal to money supply if (vi) is satisfied.

Clearly, $p = 0$ is not consistent with the incentive of sellers. Suppose $p > 0$ is an equilibrium price. An agent with $\eta \geq p$ clearly chooses to be a buyer. Thus $\eta \geq 2p$ is a transient state. Similarly, an agent with $\eta < p$ cannot buy and chooses to be a seller if $\gamma V(\eta + p) - c > \gamma V(\eta)$. Thus the measure of agents with $\eta \in [0, p)$ is equal to that with $\eta \in [p, 2p)$, since the market clearing condition is that the measure of sellers is equal to that of buyers and $\eta \geq 2p$ is a transient state. Thus the Bellman equation is as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta \in [0, p), \\ u + \gamma V(\eta - p), & \text{otherwise.} \end{cases}$$

Thus the value function is obtained as follows:

$$V(\eta) = \begin{cases} \frac{\gamma u - c}{1-\gamma^2}, & \text{if } \eta \in [0, p), \\ \frac{u - \gamma c}{1-\gamma^2}, & \text{if } \eta \in [p, 2p), \\ \vdots & \\ \frac{(1+\gamma)u - \gamma^n(u+c)}{1-\gamma^2}, & \text{if } \eta \in [np, (n+1)p), \\ \vdots & \end{cases} \tag{7}$$

Note that $\gamma V(\eta + p) - c > V(\eta)$ holds for $\eta \in [0, p)$ if

$$\gamma > \frac{1}{\theta},$$

holds. Clearly, the stationary Walrasian equilibrium allocation is that a half of agents have $\eta \in [0, p)$ and their value is $\frac{\gamma u - c}{1-\gamma^2}$, and the other half of agents have $\eta \in [p, 2p)$ and their value is $\frac{u - \gamma c}{1-\gamma^2}$. Note that the money holdings distribution is stationary if and only if $F([0, \eta]) = F([p, p + \eta])$ for any $\eta \in [0, p)$.

Theorem 2 For any $\gamma \in \left(\frac{1}{\theta}, 1\right)$, there exists a stationary Walrasian equilibrium $\langle p, F, V, \phi, T \rangle$. Moreover, any Walrasian equilibrium is characterized by

(I) V is given by (7),

(II)

$$\phi(\eta) = \begin{cases} \{\sigma\} & \text{if } \eta \in [0, p), \\ \{\beta\} & \text{if } \eta \in [p, \infty), \end{cases}$$

(III)

$$T(\eta, A) = 1 \text{ iff } \begin{cases} \eta + p \in A & \text{and} \quad \eta \in [0, p), \quad \text{or} \\ \eta - p \in A & \text{and} \quad \eta \in [p, \infty), \end{cases}$$

(IV) F satisfies $\int \eta dF = M$, $F([0, p)) = 1/2$, $F([p, 2p)) = 1/2$, and

$$F([0, \eta]) = F([p, p + \eta]), \quad \forall \eta \in [0, p).$$

Proof:
From the discussion in the above, for a given p, (V, ϕ, T) are characterized by (I), (II), and (III), and they clearly exist. Thus it is suffices to show the existence of (F, p) satisfying (IV). Let $p^* = 2M$ and F^* be such that $F^*(\{0\}) = 1/2$ and $F^*(\{p\}) = 1/2$. They clearly satisfy (IV). ∎

In the proof of the above theorem, a probability measure satisfying (IV), F^*, is presented. Of course, there are other probability measures satisfying (IV). For example, $\tilde{p} = \frac{4M}{3}$ and $\tilde{F}$ such that $\tilde{F}(\{0\}) = \tilde{F}(\{\frac{1}{2}\tilde{p}\}) = \tilde{F}(\{\tilde{p}\}) = \tilde{F}(\{\frac{3}{2}\tilde{p}\}) = \frac{1}{4}$ satisfy (IV). In the equilibrium with $(\tilde{p}, \tilde{F})$, an agent with $\frac{1}{2}\tilde{p}$ deterministically alternates between acquiring $\tilde{p}$ as a seller and spending $\tilde{p}$ as a buyer, i.e., she alternates between states $\frac{1}{2}\tilde{p}$ and $\frac{3}{2}\tilde{p}$. Thus the behavior of an agent with $\frac{1}{2}\tilde{p}$ is exactly the same as that of an agent without money. Moreover the real allocation and transactions of commodity goods on the equilibrium are completely the same as those on the equilibrium with (p^*, F^*). Indeed, the following corollary clearly holds.

Corollary 1 The distribution of value in stationary Walrasian equilibria is uniquely determined, i.e., the half of agents have $\frac{\gamma u - c}{1-\gamma^2}$, the case of $\eta \in [0, p)$, and the rest of agents have $\frac{u - \gamma c}{1-\gamma^2}$, the case of $\eta \in [p, 2p)$.

Thus in order to investigate real allocations, we can restrict our attention to the stationary Walrasian equilibrium with (p^*, F^*).[3]

[3]If we assume that there is an infinitesimally small cost of holding money, then only the equilibrium with F^* survives.

We are now ready to compare the auction markets outcome in the previous section and stationary Walrasian equilibrium outcome in this section. Setting $h_0 = 1/2$ in the value function (1) of the auction markets, we obtain $V(0) = \frac{\gamma u - c}{1-\gamma^2}$ and $V(p) = \frac{u - \gamma c}{1-\gamma^2}$. By $h_0 = 1/2$, a half of agents have the former value and the rest of agents have the latter value, i.e., exactly same as the case of the Walrasian market outcome in Corollary 1. As shown in Theorem 1, there are other stationary equilibria for $h_0 > \frac{1}{2}$, where the values are different from those in Corollary 1.

Theorem 3 The set of outcomes in the auction market equilibrium does not coincide with that of the stationary Walrasian equilibrium.

5 Real Indeterminacy of Stationary Equilibria in Monetary Models

In this section, we explore the logic behind Theorem 3. As shown in the previous sections, the auction markets have a continuum of stationary equilibria, while the Walrasian markets have essentially the unique equilibrium, and thus the outcomes do not coincide. Below, we show that there are two types of fundamental natures of monetary trades, and one has a continuum of stationary equilibrium and the other has locally unique equilibria. Of course, a typical example of the former case is the auction market and the latter case is the Walrasian market.

In the Walrasian market, the price of goods is determined in the centralized markets. Thus, as shown in Section 4, it is suffices to investigate money holdings distributions with a support expressed by $\{0, p, 2p, \ldots, Np\}$ for some positive integer N, where $p > 0$ is an equilibrium price. In auction markets, there may exist equilibrium money holdings distributions of which support are not $\{0, p, 2p, \ldots, Np\}$. However, in order to show that the outcomes in the latter case includes the outcomes in the former case, it is suffices to focus on distributions with a support expressed by $\{0, p, 2p, \ldots, Np\}$. Let $h = (h_0, h_1, \ldots, h_N)$ be a probability distribution on the support, where h_n is a measure of agent with money holding np.

Below, we compare the equilibrium conditions in the previous two sections. Focusing on the case that the measure of buyers is larger than or equal to that of sellers and

$N = 1$, the equilibrium condition for the auction markets is as follows:

$$
\begin{aligned}
p &= \frac{M}{h_1}, \\
h_0 + h_1 &= 1, \\
h_1 \frac{h_0}{h_1} &= h_0, \\
h_0 &= h_1 \frac{h_0}{h_1}, \\
V(\eta) &= \max \big\{ (-c + \gamma V(\eta + \pi(h, \xi))) \\
&\qquad \max_{b \in R_+} \nu_B(b, h, \xi) \left(u + \gamma V(\eta - \pi(h, \xi)) \right) + (1 - \nu_B(b, h, \xi)) \gamma \mathcal{V}(\eta) \big\}.
\end{aligned}
$$

On the other hand, the equilibrium condition for Walrasian markets is as follows:

$$
\begin{aligned}
p &= \frac{M}{h_1}, \\
h_0 + h_1 &= 1, \\
h_1 T(p, \{0\}) &= h_0 T(0, \{p\}), \\
h_0 T(0, \{p\}) &= h_1 T(p, \{0\}), \\
V(\eta) &= \begin{cases} \gamma V(\eta + p) - c & \text{if } \eta \in [0, p), \\ \gamma V(\eta - p) + u & \text{otherwise}, \end{cases}
\end{aligned}
$$

where the third and the forth equations are the conditions for stationarity of money holdings. In the both systems, for a given (h_0, h_1), p and V are uniquely determined by $p = \frac{M}{h_1}$ and the Bellman equations. As for (h_0, h_1), in the equilibrium condition for the auction markets, any (h_0, h_1) satisfying $h_0 + h_1 = 1$, $h_0 \geq 0$, and $h_1 \geq 0$, is a stationary distribution, since $h_1 \frac{h_0}{h_1} = h_0$ and $h_0 = h_1 \frac{h_0}{h_1}$ are identities. On the other hand, in the equilibrium condition for the Walrasian markets, (h_0, h_1) satisfying $h_0 + h_1 = 1$ is not necessarily a stationary distribution, since $h_1 T(p, \{0\}) = h_0 T(0, \{p\})$ is not an identity. (See the discussion below.) Thus there is one degree of freedom in the former system, while the solution is determinate in the latter system.

Below, we show that there are two types of fundamental natures of monetary trades, and one has a continuum of stationary equilibrium and the other has locally unique equilibria. The natures are as follows:

- the amount of money the sellers obtain is always equal to that of buyers pay even out of equilibria, and

- the amount of money the sellers obtain is not necessarily equal to that of buyers pay.

The auction market is clearly classified as the first type and the Walrasian market is classified as the second type. Indeed, in the auction markets, the amount of money the sellers obtain is ph_0 and that of buyers pay is $ph_1\frac{h_0}{h_1} = ph_0$. By this identity, any money holdings distribution is stationary, i.e., all (h_0, h_1) satisfying $h_0 + h_1 = 1$, $h_0 \geq 0$, and $h_1 \geq 0$ is stationarity, and the set of equilibria is a continuum. On the other hand, in the Walrasian markets, the amount of money the sellers obtain is ph_0 if p is large enough, i.e., $T(0, \{p\}) = 1$, and less than ph_0 otherwise, and that of buyers pay is ph_1 if p is small enough, i.e., $T(1, \{0\}) = 1$, and less than ph_1 otherwise. Thus they are not necessarily the same.

In order to understand the logic of indeterminacy, we investigate a more general case. That is, in the auction markets, we consider the case that some agents with np becomes sellers and the rest of them become buyers. Then the transition probability is represented by $Q_n^S \geq 0$ and $Q_n^B \geq 0$ such that $Q_n^S + Q_n^B = 1$, where Q_n^S is the measure of agents who have np and become sellers and Q_n^B the measure of agents who have np and become buyers. Then the stationarity of money holdings is written as:

$$\begin{aligned}
D_0 &\equiv Q_1^B - Q_0^S = 0, \\
D_1 &\equiv Q_2^B + Q_0^S - Q_1^B - Q_1^S = 0, \\
&\vdots \qquad\qquad \vdots \\
D_N &\equiv Q_{N-1}^S - Q_N^B = 0.
\end{aligned}$$

In this case, the amount of money the sellers obtain is $\sum_{n=0}^{N-1} pQ_n^S$ and the amount of money the buyers pay is $\sum_{n=1}^{N} pQ_n^B$, and they are identically the same because of the rule of the uniform price auction. Thus

$$\begin{aligned}
\sum_{n=0}^{N} npD_n &= 0pD_0 + 1pD_1 + 2pD_2 + \cdots + NpD_N \\
&= \sum_{n=0}^{N-1}((n+1)pQ_n^S - npQ_n^S) + \sum_{n=1}^{N}((n-1)pQ_n^B - npQ_n^B) \\
&= \sum_{n=0}^{N-1} pQ_n^S - \sum_{n=1}^{N} pQ_n^B \\
&= 0
\end{aligned}$$

is an identity. Moreover, each Q_n^S (Q_n^B) appears twice in the above equations as a positive term and a negative term. Thus another identity

$$\begin{aligned}\sum_{n=0}^{N} D_n &= D_0 + D_1 + D_2 + \cdots + D_N \\ &= 0\end{aligned}$$

holds. If $D_2 = D_3 = \cdots = D_N = 0$ holds, then by the above two identities, $D_1 = 0$ and $D_0 = 0$ follow. Thus among $D_n = 0, n = 0, 1, \ldots, N$, two equations are redundant. Since the number of variables $(h_0, h_1, \ldots, h_N)$ is $N+1$ and the number of linearly independent equations including $\sum_{n=0}^{N} h_n = 1$ is N, there is at least one degree of freedom. Especially, in the case of $N = 1$ in Section 3, both $D_0 = 0$ and $D_1 = 0$ are redundant and any (h_0, h_1) satisfying $h_0 + h_1 = 1$ is a stationary distribution. Note that an exmmple with $N = 2$ is given in Section A in Appendix.

One might think that the indeterminacy of equilibria in the auction market model is related to indivisibility of goods or rationing. However, from the above arguments, it is easy to see that the indeterminacy has nothing to do with indivisibility of goods or rationing. Indeed, the identity $\sum_{n=0}^{N} pnD_n = 0$ holds whenever monetary trades is of the first type.

In decentralized market models with divisible money, monetary trades typically classified as the first type. (See, for example, Green and Zhou [8], Kamiya and Shimizu [13], Matsui and Shimizu [15], and Zhou [23].) For example, suppose (a) a buyer randomly meets a seller, (b) then the buyer offers a price and an amount of good she wants to buy, and (c) finally the seller decides whether to accept or to reject the offer. In this case, in each matching the amount of money the buyer pays is always the same as that of the seller obtains even out of equilibria. Thus in the economy the total amount of money the buyers pay is always the same as that of the sellers obtain even out of equilibria. By the same logic as in the centralized economy, the equilibria are typically indeterminate. The only one difference between decentralized market models and centralized market models of the first type is that the prices of goods in matchings might be different in decentralized market models. (See Kamiya and Sato [12] and Matsui and Shimizu [15].)

Appendix

A Mixed Strategy Equilibria in Centralized Auction Markets

In this section, we prove the existence of mixed strategy stationary equilibria in the centralized auction market model investigated in Section 3. First, we define a mixed strategy stationary equilibrium. A candidate for an equilibrium mixed strategy can be defined as a function of a money holding $\eta \in R_+$,

$$\tilde{\xi} : R_+ \to \Omega\left(\{\sigma\} \cup (\{\beta\} \times R_+) \cup \{\nu\}\right),$$

where $\Omega(A)$ is the set of probability distributions on a set A. In parallel with Definition 1 in Section 3, a mixed strategy stationary equilibrium is defined in the case of a finite number of bid prices.

We consider the following mixed (behavioral) strategy:

- an agent with $\eta \in [0, p)$ chooses to be a seller,
- an agent with p chooses to be a buyer and bids p with probability $\delta \in (0, 1)$, and chooses to be a seller with probability $1 - \delta$, and
- an agent with $\eta \in (p, \infty)$ chooses to be a buyer and bids η.

Note that under the above strategy, a money holding $2p$ is no longer a transient state. Let $h = (h_0, h_1, h_2)$ and

$$r = \frac{h_0 + (1-\delta)h_1 - h_2}{\delta h_1}.$$

Note that an agent with p can buy a good with probability r. It will be shown that $r \in [0, 1]$ in equilibria. Under the above strategy, the stationarity of money holding distribution can be written as follows:

$$\begin{aligned}
&r\delta h_1 = h_0,\\
&rh_2 + h_0 = r\delta h_1 + (1-\delta)h_1,\\
&rh_2 = (1-\delta)h_1,\\
&h_0 + h_1 + h_2 = 1.
\end{aligned}$$

Note that, as we have shown in Section 5, two of the first three equations are redundant. The following stationary distribution is obtained as follows:

$$h_0 = \frac{\delta r}{2 - \delta + \delta r},\ h_1 = \frac{1}{2 - \delta + \delta r},\ h_2 = \frac{1-\delta}{2 - \delta + \delta r}. \tag{8}$$

Note that $r \geq 0$ implies $h_n \in [0,1]$ for $n = 0,1,2$.

If $\delta > 0$, the uniform price is p. It follows that the Bellman equation is the same as (1) in Section 3, though the definition of r is different, and then we obtain the same value function. The incentive conditions for choosing the above strategy are as follows:

(i) $V(0) \geq 0$,

(ii) $V(p) = -c + \gamma V(2p)$.

The first inequality is the incentive to be a seller for an agent without money. Since in (1) $V(p)$ is defined as the value when an agent becomes a buyer, the second inequality implies that the agent is indifferent between a buyer and a seller. As in Section 3, the other incentive conditions can be easily checked. Sufficient conditions for the above inequalities are (2) and (4) in Section 3, where the definition of r is different, and

$$r = \frac{\gamma\theta - 1}{(1+\gamma-\gamma^2)\theta - \gamma^2}, \tag{9}$$

which is obtained from (ii). Note that $r \in [0,1]$ follows from (4). It is also worthwhile noting that, substituting (9) into (8), (h_0, h_1, h_2) can be parametrized by δ.

Then substituting (9) into (2), we obtain the following inequality by (4):

$$-(\theta+1)\gamma^3 + (\theta^2+\theta+1)\gamma^2 - (\theta-1)\gamma - \theta \geq 0.$$

It is verified that there exists $\underline{\gamma} \in \left(\frac{1}{\theta}, 1\right)$ such that the above inequality holds for any $\gamma \in \left(\underline{\gamma}, 1\right)$.

Theorem 4 For any given $\gamma \in \left(\underline{\gamma}, 1\right)$, the above mixed strategy is a stationary equilibrium with a discrete money holdings distribution for any $\delta \in (0,1]$.

Note that since (h_0, h_1, h_2) and (V_0, V_1, V_2) depend on δ, there is real indeterminacy in equilibria.

B Relaxing the Limited Participation Constraint

In this section we show that the limited participation constraint, an agent can join only one market in a period, is not crucial for the results in Sections 3 and 4 . To see this, in this section, we assume that agents can sell and buy in the same period. Below, under the assumption, we show the real indeterminacy of equilibria in the decentralized auction markets and the real determinacy of equilibria in the Walrasian markets.

B.1 Centralized Auction Markets

First, we analyze centralized auction markets. We focus on stationary equilibria in which money holdings distribution has support $\{0, p\}$. We investigate the following strategy:

- an agent with $\eta \in (0, p]$ chooses to be only a seller,
- an agent with p chooses to be only a buyer and bids p,
- an agent with $\eta \in (p, 2p)$ chooses to be a buyer and seller, and as a buyer, bids η, and
- an agent with $\eta \in [2p, \infty)$ chooses to be only a buyer and bids η.

Under the strategy, the value function is expressed as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta \in [0, p), \\ r\left(u + \gamma V(0)\right) + (1 - r)\gamma V(p), & \text{if } \eta = p, \\ u - c + \gamma V(\eta), & \text{if } \eta \in (p, 2p), \\ u + \gamma V(p)t, & \text{if } \eta \in [2p, \infty). \end{cases}$$

where

$$r = \frac{h_0}{1 - h_0}.$$

Let n be the integer part of η and ι be the residual, then η is uniquely expressed as $\eta = np + \iota$. Thus the value function becomes as follows:

$$V(np + \iota) = \begin{cases} \frac{r\gamma u - (1-\gamma+r\gamma)c}{(1-\gamma)(1+r\gamma)}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{1+r\gamma}\left[(1 - r + r\gamma)u + r\gamma c\right]\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{\gamma u - c}{1-\gamma}, & \text{if } \iota \neq 0,\ n = 0, \\ \frac{u - \gamma^{n-1}c}{1-\gamma}, & \text{if } \iota \neq 0,\ n \neq 0. \end{cases}$$

We consider the case of $\gamma > \frac{1}{\theta}$. Then the incentive for an agent with $\eta = np + \iota$ and $\iota \neq 0$ is easily verified. The other incentive conditions are as follows:

(i) incentive for an agent with p to be only a seller instead of doing nothing,

(ii) incentive for an agent with p to be only a buyer instead of being only a seller,

(iii) incentive for an agent with p to be only a buyer instead of being a buyer and seller, and

(iv) incentive for an agent with np where $n \geq 2$ to be only a buyer instead of being a buyer and seller.

Since the value of $\eta = np$ is the same as the one in Section 3, a sufficient condition for (i) and (ii) is also (2) and (5). Moreover, the proof of Theorem 1 shows that (2) is satisfied if (5) and $\gamma \geq \frac{1}{\theta}$ hold.

(iii) is expressed by

$$V(p) \geq r\left[u - c + \gamma V(p)\right] + (1-r)\left[-c + \gamma V(2p)\right],$$

and it is equivalent to

$$-\left[\gamma(2-\gamma)\theta - \gamma^2\right] r^2 + \left[\gamma(2-\gamma)\theta - \gamma^2\right] r - (\gamma\theta - 1) \geq 0. \tag{10}$$

Denote the LHS by $F(r)$. Then it is verified that $F(0) = F(1) < 0$ and the coefficient of r is negative. Therefore F has the maximum value at $r = \frac{1}{2}$. We obtain

$$F\left(\frac{1}{2}\right) = \frac{1}{4}\left[-(\theta+1)\gamma^2 - 2\theta\gamma + 4\right].$$

Since $\gamma > \frac{1}{\theta}$ and $F\left(\frac{1}{2}\right)$ is positive for $\gamma \in \left(-2, \frac{2}{\theta+1}\right)$, we fix a $\gamma \in \left(\frac{1}{\theta}, \frac{2}{\theta+1}\right)$. Then, since $F\left(\frac{1}{2}\right) > 0$, we can find $\epsilon_1 > 0$ such that (10) holds for any $r \in \left[\frac{1}{2} - \epsilon_1, \frac{1}{2} + \epsilon_1\right]$.

Next, (iv) is expressed by

$$V(np) \geq u - c + \gamma V(np), \quad n \geq 2,$$

and it is equivalent to

$$-r(\theta+1)\gamma^2 + \left[r(\theta+1) - \theta\right]\gamma + 1 \geq 0.$$

Denote the LHS by $G(\gamma)$. Then it is verified that $G(0) > 0$, $G(1) < 0$, and $G'(\gamma) < 0$ for any $\gamma \in (0,1)$. Thus we obtain

$$G\left(\frac{2}{\theta+1}\right) = \frac{\theta-1}{\theta+1}(2r-1).$$

Since $\gamma \in \left(\frac{1}{\theta}, \frac{2}{\theta+1}\right)$, we can find $\epsilon_2 > 0$ such that the above inequality holds for any $r \in [\frac{1}{2} - \epsilon_2, 1]$. Note that, by $\gamma \in \left(\frac{1}{\theta}, \frac{2}{\theta+1}\right)$, $r \geq \frac{1}{\theta+4}$ is sufficient for (5).

Finally, let $\overline{\epsilon} = 2\epsilon_1$ and $\underline{\epsilon} = 2\min\left\{\epsilon_1, \epsilon_2, \frac{\theta+2}{2(\theta+4)}\right\}$. Then by the above discussion there exists a stationary equilibrium with the specified strategy for any $r \in \left[\frac{1}{2}(1-\underline{\epsilon}), \frac{1}{2}(1+\overline{\epsilon})\right]$. Note that $r \in \left[\frac{1}{2}(1-\underline{\epsilon}), \frac{1}{2}(1+\overline{\epsilon})\right]$ is equivalent to $h_0 \in \left[\frac{1-\underline{\epsilon}}{3-\underline{\epsilon}}, \frac{1+\overline{\epsilon}}{3+\overline{\epsilon}}\right]$.

Theorem 5 For any given $\gamma \in (\frac{1}{\theta}, \frac{2}{\theta+1})$, there exist $\underline{\epsilon} > 0$ and $\overline{\epsilon} > 0$ such that there exists a stationary equilibrium with a discrete money holdings distribution for any $h_0 \in \left[\frac{1-\underline{\epsilon}}{3-\underline{\epsilon}}, \frac{1+\overline{\epsilon}}{3+\overline{\epsilon}}\right]$.

B.2 Walrasian Markets

Next, we consider Walrasian markets. We consider the same model as in Section 4 besides C and ϕ are defined as $C = \{(1,1),(1,0),(0,1),(0,0)\}$ and $\phi : R_+ \to \{\omega, \beta, \sigma, \nu\}$ where ω stands for 'sell and buy'. Note that the Bellman equation is also expressed by (6).

Below, we consider the case that $\gamma > 1/\theta = c/u$. As in the discussion in Section 4, an equilibrium price, if it exists, is $p > 0$. Since an agent with $\eta \in [0, p)$ cannot buy,

$$V(\eta) = \max\{0, -c + \gamma V(\eta + p)\}$$

holds. Since an agent with $\eta + p$ can buy and sell,

$$V(\eta + p) \geq \frac{u - c}{1 - \gamma},$$

holds. Thus by

$$-c + \gamma \frac{u - c}{1 - \gamma} = \frac{\gamma u - c}{1 - \gamma} > 0,$$

we obtain

$$V(\eta) = -c + \gamma V(\eta + p), \tag{11}$$

i.e., $\phi(\eta) = \{\sigma\}$. By (11),

$$u - c + \gamma V(\eta) = u - c + V(\eta - p) + c > u + \gamma V(\eta - p).$$

holds for any $\eta \in [p, 2p)$. Since it is easily verified that $\sigma \notin \phi(\eta)$, we obtain $\phi(\eta) = \{\omega\}$. Moreover, it is also easily verified that $\phi(\eta) = \{\beta\}$ for any $\eta \in [2p, \infty)$. Then we obtain the following result.

Theorem 6 For any $\gamma \in \left(\frac{1}{\theta}, 1\right)$, there exists a stationary Walrasian equilibrium $\langle p, F, V, \phi, T \rangle$. Moreover, any Walrasian equilibrium is characterized by

(i)

$$\phi(\eta) = \begin{cases} \{\sigma\} & \text{if } \eta \in [0, p), \\ \{\omega\} & \text{if } \eta \in [p, 2p), \\ \{\beta\} & \text{if } \eta \in [2p, \infty), \end{cases}$$

(ii) V is defined as

$$V(\eta) = \begin{cases} \frac{\gamma u - c}{1-\gamma}, & \text{if } \eta \in [0, p), \\ \frac{u-c}{1-\gamma}, & \text{if } \eta \in [p, 2p), \\ \vdots & \\ \frac{u - \gamma^{n-1} c}{1-\gamma}, & \text{if } \eta \in [np, (n+1)p), \\ \vdots & \end{cases}$$

(iii)

$$T(\eta, A) = 1 \text{ iff } \begin{cases} \eta + p \in A & \text{and} \quad \eta \in [0, p), \quad \text{or} \\ \eta \in A & \text{and} \quad \eta \in [p, 2p), \quad \text{or} \\ \eta - p \in A & \text{and} \quad \eta \in [2p, \infty), \end{cases}$$

(iv) F satisfies

$$F([p, 2p)) = 1,$$
$$\int_p^{2p} \eta dF = M.$$

Corollary 2 The distribution of value in stationary Walrasian equilibria is uniquely determined, i.e., the value of any agent is $\frac{u-c}{1-\gamma}$.

The above corollary implies that the indeterminacy is not a real one but a nominal one as in Section 4.

C Decentralized Auctions

In the same environment as in Section 2, we consider an economy, where trades take place in decentralized second price auction markets, and show that there is also a continuum of stationary equilibria.

In each period, each agent first chooses whether to be a seller or a buyer (or doing nothing). Each seller posts a minimum bid of her second-price auction.[4] After observing the distribution of posted minimum bids, each buyer simultaneously chooses which auction he participates in. After observing the number of the other participants in the auction he participates in and bids a price.

In the environment we have described, we consider the following strategy:

[4]Nothing would change if we assume the sellers also choose her auction format.

- an agent with $\eta \in [0, p)$ chooses to be a seller and post a minimum bid p,
- an agent with p chooses to be a buyer and always bids p,
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer, and
  - bids p if there is no participant in the auction,
  - bids η if there are other participants in the auction.

Moreover, we consider the stationary equilibria in which

(i) the support of the money holdings distribution is $\{0, p\}$,

(ii) $h_0 \leq \frac{1}{2}$, and

(iii) every buyer randomizes with equal fractions between auctions with the same minimum bids.

The equilibrium with (iii) is often investigated in many directed search model, e.g., Peters [17]. In such an equilibrium, while the measure of the sellers is no less than that of the buyers, there might be the sellers whom no buyer visit. To be more precise, let $\rho = \frac{1-h_0}{h_0}$ (then $\rho \geq 1$), and the probability a seller succeeds to sell a good in each period is defined as $1 - e^{-\rho}$, and the probability each buyer with p succeeds to buy a good in each period is defined as $\frac{1-e^{-\rho}}{\rho}$.[5] Denote by α the former probability.

The value function is defined as follows:

$$V(\eta) = \begin{cases} \alpha\left(-c + \gamma V(\eta + p)\right) + (1 - \alpha)\gamma V(\eta), & \text{if } \eta < p, \\ \frac{\alpha}{\rho}\left(u + \gamma V(0)\right) + \left(1 - \frac{\alpha}{\rho}\right)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p. \end{cases}$$

Then, letting $\eta = np + \iota$ with an integer n and the residual ι, we obtain

$$V(np + \iota) = \begin{cases} \frac{\gamma\alpha^2(u-c)-(1-\gamma)\alpha\rho c}{(1-\gamma)[(1-\gamma)\rho+\gamma\alpha(1+\rho)]}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}u - \gamma^{n-1}\frac{\left[-\gamma\alpha^2+\alpha(-1+2\gamma+\gamma\rho)+(1-\gamma)\rho\right]u+\gamma\alpha^2 c}{(1-\gamma)[(1-\gamma)\rho+\gamma\alpha(1+\rho)]}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{1}{1-\gamma}u - \gamma^n\frac{u+\alpha c}{(1-\gamma)(1+\gamma\alpha)}, & \text{if } \iota \neq 0. \end{cases}$$

The incentive conditions are the same as in the centralized auction model. In other words,

(I) $V(\iota) \geq 0$ for any $\iota \in [0, p)$.

[5] We obtain these probabilities in the limit of large but finite economies. See Peters [17] for the details.

(II) $V(np+\iota) \geq -c + \gamma V((n+1)p+\iota)$ for any $n \geq 1$ and $\iota \in [0,p)$.

As for (I),

$$\gamma \geq \underline{\gamma}(\rho) = \frac{\rho}{(\theta-1)\alpha+\rho},$$

is a sufficient condition. Also as for (II),

$$\rho(1-\gamma\theta)(1-\gamma+\gamma\alpha) + (\theta-1)\gamma^2\alpha^2 + \theta(1-\gamma) \geq 0,$$

is a sufficient condition. Furthermore, it is equivalent to

$$\gamma \leq \overline{\gamma}(\rho) = \frac{\rho(1-\alpha+\theta)+\theta-T}{2\left[\rho\theta(1-\alpha)+(\theta-1)\alpha^2\right]},$$

where

$$T^2 = (\theta-1+\alpha)^2\rho^2 + 2\left[-\theta(\theta-1+\alpha) + 2\alpha(\theta^2-\theta\alpha+\alpha)\right]\rho + \left[\theta^2 - 4\theta(\theta-1)\alpha^2\right].$$

After some tedious calculation, it is verified

$$\begin{aligned} &\underline{\gamma}(1) < \overline{\gamma}(1), \\ &|\underline{\gamma}'(1)| < \infty, \\ &|\overline{\gamma}'(1)| < \infty, \end{aligned}$$

then there exists a region of parameters such that there is a continuum of stationary equilibria.

Theorem 7 There exist $0 < \underline{\gamma} < \overline{\gamma} < 1$ and $\overline{h}_0 \in (0, \frac{1}{2})$ such that, for any given $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there exists a stationary equilibrium for any $h_0 \in \left(\overline{h}_0, \frac{1}{2}\right]$.

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