Voluntarily Repeated Prisoner's Dilemma with Reference Letters *

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Incomplete and Very Premilinary
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1 Basic Model

1.1 Extended Prisoner's Dilemma and Its Extensions

We consider a continuum of homogenous players. In each period, $(1-\delta) \in (0,1)$ ratio of players die and are replaced by newly born players of the same ratio. Players do not discount future except the possibility of their own death (i.e., their innate discount rate is zero.) It follows that each player expects to live for $\bar{L} = \frac{1}{1-\delta}$ days from now.

We modify our basic model of EPD by adding following two extra features:

- Players who arrive in the matching pool may have a *reference letter* depending upon the reason why she comes to the matching pool.
  - Whether or not a partner has a reference letter (denoted by index $r \in \{Y, N\}$ where $r = Y$ indicates the fact that she has a reference letter and $r = N$ no reference letter) becomes common knowledge between partners at the beginning of the partnership.
  - In this paper, we model the mechanism of reference letter endowment being exogenous. That is, we assume that whether or not a player arriving in the

*This is a very incomplete and preliminary summary of our ongoing research which is an extension of our previous paper ("Voluntarily Repeated Prisoner's Dilemma," by Takako Fujiwara-Greve and Masahiro Okuno-Fujiwara (hereafter denoted as GO2005.)

matching pool has a reference letter is determined solely by the reason why she comes to the pool and this rule is pre-determined.

  - Player can condition their strategy upon this information $(r, r') \in \{Y, N\}^2$.

- Player can take extra action $z \in \{form, skip\}$ (to which we will refer as a *partnership formation decision*) when she is matched with a potential partner in the matching pool. If $z = form$, player forms a partnership with the matched player as in the basic model. If $z = skip$, player skips to form partnership in this period (remains without partner and receives payoff of 0 in this period) and will match with another potential partner in the matching pool next period.

1.2 Strategies

Action history of a partnership at τ is a $(\tau - 1)$-tuple of action pairs $h_\tau := (x_t, x'_t)_{t=1}^{\tau-1} \in (\{C, D\} \times \{C, D\})^{\tau-1} := H_\tau$, with usual convention of $H_1 := \{\emptyset\}$. A set of all partnership play histories is $H := \cup_{t=1}^{\infty}$.

A *partnership strategy* s prescribes player's partnership formation decision, play action and continuation decision as functions of (own and partner's) reference letter posession $(r, r') \in \{Y, N\}^2$ and partnership history $h_t \in H$. That is, $s = (z, (x_t, y_t)_{t=1}^{\infty})$ such that:

(1) $z : \{Y, N\}^2 \mapsto \{form, skip\}$ prescribes to form a partnership with the matched player if $z(r, r') = form$ and to skip otherwise.

(2) conditional upon the partnership being formed and being still active at the beginning of t:

  (a) $x_t : \{Y, N\}^2 \times H_t \mapsto \{C, D\}$ prescribes action choice at t as $x_t(r, r'; h_t) \in \{C, D\}$, and

  (b) $y_t : \{Y, N\}^2 \times H_t \times \{C, D\}^2 \mapsto \{k, e\}$ prescribes continuation decision at the end of t as $y_t(r, r'; h_t, \boldsymbol{x}_t) \in \{k, e\}$ where $\boldsymbol{x}_t = (x_t, x'_t)$.

We denote the set of partnership strategies of the reference letter model by S. A player in the matching pool is characterized completely by her strategy $s \in S$ and whether or not she has a reference letter $r \in \{Y, N\}$. A player is, then, characterized by a pair (s, r), to which we shall refer as her *position*.

1.3 Simple Strategies

There are numerous strategies which are potentially available. Partly because we want to make our analysis simpler and partly because we believe players in the real world can use only those strategies which are relatively simple, following GO2005, we restrict our attention to the following simple strategies.

Acceptable Path Let $\Omega := H_\infty \cup \{\emptyset\}$ be the set of **potentially acceptable paths**. $H_\infty := (\{C, D\} \times \{C, D\})^\infty$ denotes a set of all infnite action profile paths, where an infinite action profile path (of a partnership) is $q = (q(t))_{t=1}^\infty \in H_\infty$ with $q(t)$ being an action pair $q(t) := (q_1(t), q_2(t)) \in \{C, D\}^2$. [1]

Intended interpretation is that a strategy is simple if:

- It specifies an acceptable action profile path for any reference letter position $(r, r') \in \{Y, N\}^2$.
- If acceptable path $q \in H_\infty$, choose $form$ decision and establish partnership with matched player and,
  - player keeps the partnership if and only if the partnership's play history is consistent with q, and
  - as long as partnership is maintained in t, choose the play action $q_1(t)$ prescribed by q.
- If it is $\emptyset$, choose *skip* decision and remain without partner for the period.

Definition 1.1. A strategy $s \in S$ is a *simple strategy with acceptable paths* Q if there exists $Q : \{Y, N\}^2 \mapsto \Omega$.

The set of all simple strategies is denoted as $\mathcal{Q}$. $(r, Q) \in \{Y, N\} \times \mathcal{Q}$ describes *status* of a player at certain (calender) time.

1.4 Examples

Define:

- $q_0 = ((C, C), (C, C), \dots)$
- $q_T = (\overbrace{(D, D), \dots, (D, D)}^{T\ times}, (C, C), (C, C), \dots)$

Example 1: Strategy $Q_T^{Babbling}$

Play c_T regardless of reference letter possession.

- $Q(r, r') = \{q_T\}$ fpr all $(r, r') \in \{Y, N\}^2$.

Example 2: Strategy Q_T^{RL}

Play c_0 when $(r, r') = (Y, Y)$, play c_T otherwise.

- $Q(Y, Y) = \{q_0\}$,
- $Q(r, r') = \{q_T\}$ if $(r, r') \neq (Y, Y)$.

[1] In GO2005, we allowed finite action profile path to be acceptable. For the sake of simplicity, in this paper we assume that acceptable action profile paths must be of infinite length.

Example 3: Picky Strategy Q_T^{Picky}

Play c_0 when $(r, r') = (Y, Y)$, play *skip* when $(r, r') = (Y, N)$,
play c_T if $(r, r') = (N, Y)$ or $(r, r') = (N, N)$.

- $Q(Y, Y) = \{q_0\}$,
- $Q(Y, N) = \emptyset$,
- $Q(N, r') = \{q_T\}$ for any r'.

1.5 Partnership Payoffs

Consider a player in a partnership, whose status is $(r, Q) \in \{Y, N\} \times \mathcal{Q}$ and the partner' status is (r', Q'). As usual, we can define:

- *average payoff* $v^I(r, Q; r', Q')$ that player with status (r, Q) expects to receive during the partnership with (r', Q'), and
- *expected length of partnership* $L(r, Q; r', Q')$ that player with (r, Q) expects to spend with (r', Q').

That leaves $\bar{L} - L(r, Q; r', Q')$ days for which this players expects to live after this partnership breaks up. Clearly, there are two distinct ways in which the parntnership beaks up;

- after partnership breaks up, the player goes to the matching pool with $r = Y$, and
- after the breakup, the player goes to the matching pool with $r = N$.

In order to differntiate these two possibilities, we divide $\bar{L} - L(r, Q; r', Q')$ into $L^Y(r, Q; r', Q')$ and $L^N(r, Q; r', Q')$ with $L(r, Q; r', Q') = L^Y(r, Q; r', Q') + L^N(r, Q; r', Q')$.

- $L^Y(r, Q; r', Q')$ denotes the expected length of periods for which this player expects to spend, after this partnership, by first visiting the matching pool with the status (Y, Q), while
- $L^N(r, Q; r', Q')$ denotes the expected length of periods for which this player expects to spend, after this partnership, by first visiting the matching pool with the status (N, Q).

When "skip" occurs If either $Q(r, r') = \emptyset$ or $Q'(r', r) = \emptyset$, then $v^I(r, Q; r', Q') = v^I(r', Q'; r, Q) = 0$ and $L(r, Q; r', Q') = L(r', Q'; r, Q) = 1$ because, in this case, partnership is not formed and players remain unmatched for one period.

1.6 Status Distribution and Payoffs in Matching Pool

State of matching pool is described by a player status distribution $p \in \mathcal{P}(S \times \{Y, N\})$ where $p(r, Q)$ denotes the proportion of players of the status $(r, Q) \in \mathcal{Q} \times \{Y, N\}$ in the matching pool.

Average payoff that a player expects to receive, upon entering matching pool but before getting matched with a particular partner, depends upon matching pool status distribution and her own status (r, Q). Formally, they are, for $r = Y, N$:

$$\bar{L} \cdot v(r, Q; p) = \sum_{(r', Q')} [L(r, Q; r', Q')v^I(r, Q; r', Q') + L^Y(r, Q; r', Q')v(Y, Q; p) + L^N(r, Q; r', Q')v(N, Q; p)]. \tag{1}$$

1.7 Reference Letter Assignment

There are six different reasons why a player happens to be in the matching pool. They are:

(a) Player is newly born and comes to the matching pool for the first time,

(b) Her previous partner died before trust is established (i.e., partner died in TB periods),

(c) Her previous partner died after trust is established (i.e., partner died in CP periods),

(d) Her previous partner took an action which is unacceptable for her,

(e) She took an action which was unacceptable for her previous partner,

(f) She and her partner agreed to break up the partnership.

For each of these occasions, reference letter may be issued by her previous partner (or his bereaved family). There are two questions that we must settle before we make assumptions about when a reference letter is issued for her.

Authenticity of reference letter First is the question of letter's authenticity. A player who comes to the matching pool can always write a fake letter for herself even if she has no authentic letter. For this paper, we assume that her new partenr can always check letter's authenticity with its issuer. Still, any pair of players, neither holding reference letter, may agree to form a partnership for one period and depart by writing "authentic" letters to each other.

In order to avoid the difficulty that may arise with these fake, and nearly fake, letters, in this paper we assume that reference letter is actually a "vitae" and contains personal history of reference letter possession. Thus, in this paper, players can check trust history of a player with any of her past partners (or their bereaved family members).

Incentive to issue reference letter Second, we need to check whether or not her previous partner has an incentive to issue such a letter. Note that, in our model, hodling reference letter is not important but holding “no” reference letter is, because having no letter may imply that the player is not trustworthy. Put differently, issuing no reference letter is a part of sanction against deviation to the norm. If her partner deviates the norm to which she subscribes, she sanctions her partner not only by terminating the partnership but also issuing no reference letter.

With such interpretations, it seems harmless to assume that letter is issued in cases (c) and (f). (However, as we described in the previous subsection, letters issued in case (f) have no substance in the model of this paper.) On ther other hand, we can assume that letter is not issued in cases (d) and (e). This leaves cases (a) and (b).

School teachers and/or family friends often write reference letters for new school graduates and new comers to job markets. Nonetheless, new graduates and/or job market newcomers do not have previous experiences and the quality of such letters are likely to be inferior compared with those letters who are written by her previous employers.

Moreover, from analytical viewpoints, we need a continuous inflow of players without reference letter in order to have a stationary matching pool distribution with positive measure of players without reference letter. Hence, we assume (at least for this version of the paper) players of type (a) carry no reference letter.

In case (b), whether or not she obtains letter is likely to depend upon the starting condition of previous partnership. If it started with both partners having letters, then it is reasonable to assume that the bereaved family issues the letter. On the other hand if at least one partner started without letter, it is reasonable to assume that letters are not issued.

2 Examples of NSD with reference letter

2.1 Example 1

Consider a monomorphic strategy distribution $p_T \in \mathcal{P}(S \times \{Y, N\})$ that consists of the following strategy Q (babbling strategy in example 1 in section 1.4).

- $Q(r, r') = \{q_T\}$ for all $(r, r') \in \{Y, N\}^2$.

This strategy distribution takes reference letter as a payoff irrelevant signal and ignores it. Hence, the result of this strategy is equivalent with the result of a monomophic NSD in our “No Reference Letter (NRL)” model. Huts, if $T \geq \underline{\tau}(\delta, G)$, p can be sustained as a NSD as a kind of ”babbling equilibrium$''$ of our reference letter model.

To be more precise, average payoff for a player who enters the matching pool is:

$$v^M_{NRL} = (1-\delta^{2T})d + \delta^{2T}c = c + (1-\delta^{2T})(c-d),$$

while average payoff (for the rest of her life) for a player who is in the cooperation period is:

$$v^{CP}_{NRL} = \frac{\frac{c}{1-\delta^2} + [\frac{1}{1-\delta} - \frac{1}{1-\delta^2}]v^M_{NRL}}{\frac{1}{1-\delta}} = \frac{1}{1+\delta}c + \frac{\delta}{1+\delta}v^M_{NRL}.$$

It follows that the BR condition is written as:

$$\begin{aligned}
g-c &\leq \frac{\delta^2}{1-\delta}[v^{CP}_{NRL} - v^M_{NRL}] + \frac{\delta(1-\delta)}{1-\delta}[v^M_{NRL} - v^M_{NRL}] \\
&= \frac{\delta^2}{1-\delta}\left\{\frac{1}{1+\delta}[c - v^M_{NRL}] + \frac{\delta}{1+\delta}[v^M_{NRL} - v^M_{NRL}]\right\} + \frac{\delta(1-\delta)}{1-\delta}[v^M_{NRL} - v^M_{NRL}] \\
&= \frac{\delta^2}{1-\delta^2}[c - v^M_{NRL}] + \frac{\delta}{1-\delta^2}[v^M_{NRL} - v^M_{NRL}] \\
&= \frac{\delta^2}{1-\delta^2}(1-\delta^{2T})(c-d) := TC^{NRL}, \qquad (2)
\end{aligned}$$

because, even if the player does not deviate, with probability $\delta(1-\delta)$, she will go back to the matching pool next period because her partner dies but she survives.

2.2 Example 2

Consider the following strategy Q (strategy of Example 2 in subsection 1.4).

- $Q(Y,Y) = \{q_0\}$,
- $Q(r,r') = \{q_T\}$ if $(r,r') \neq (Y,Y)$.

We check if a monomorphic strategy distribution $p \in \mathcal{P}(Q \times \{Y,N\})$ can be a NSD with appropriate parameter values. For the brevity's sake, we write $p(Q,Y) = \alpha$ and $p(Q,N) = 1-\alpha$.

We first compute the average payoff of player with r who is to begin a day in the matching pool, which should be denoted as $v(r,Q;\alpha)$. For the brevity's sake, we shall suppress Q from notations because we will be concerned with a monomorphic distribution. Thus, for example, we shall denote $v(r;\alpha)$ instead of $v(r,Q;\alpha)$.

By definition:

$$\begin{aligned}
\bar{L}v(Y;\alpha) &= \alpha[L(Y,Y)v^I(Y,Y) + L^Y(Y,Y)v(Y;\alpha) + L^N(Y,Y)v(N;\alpha)] \\
&\quad +(1-\alpha)[L(Y,N)v^I(Y,N) + L^Y(Y,N)v(Y;\alpha) + L^N(Y,N)v(N;\alpha)], \\
\bar{L}v(N;\alpha) &= \alpha[L(N,Y)v^I(N,Y) + L^Y(N,Y)v(Y;\alpha) + L^N(N,Y)v(N;\alpha)] \\
&\quad +(1-\alpha)[L(N,N)v^I(N,N) + L^Y(N,N)v(Y;\alpha) + L^N(N,N)v(N;\alpha)],
\end{aligned}$$

or:

$$\begin{bmatrix} L^{YY}(\alpha) & L^{YN}(\alpha) \\ L^{NY}(\alpha) & L^{NN}(\alpha) \end{bmatrix} \begin{bmatrix} v(Y;\alpha) \\ v(N;\alpha) \end{bmatrix} = \begin{bmatrix} V^I(Y;\alpha) \\ V^I(N;\alpha) \end{bmatrix},$$

where:

$$\begin{aligned}
\begin{bmatrix} L^{YY}(\alpha) & L^{YN}(\alpha) \\ L^{NY}(\alpha) & L^{NN}(\alpha) \end{bmatrix} &= \begin{bmatrix} \bar{L} - [\alpha L^Y(Y,Y) + (1-\alpha)L^Y(Y,N)] & -[\alpha L^N(Y,Y) + (1-\alpha)L^N(Y,N)] \\ -[\alpha L^Y(N,Y) + (1-\alpha)L^Y(N,N)] & \bar{L} - [\alpha L^N(N,Y) + (1-\alpha)L^N(N,N)] \end{bmatrix}, \\
\begin{bmatrix} V^I(Y;\alpha) \\ V^I(N;\alpha) \end{bmatrix} &= \begin{bmatrix} \alpha L(Y,Y)v^I(Y,Y) + (1-\alpha)L(Y,N)v^I(Y,N) \\ \alpha L(N,Y)v^I(N,Y) + (1-\alpha)L(N,N)v^I(N,N) \end{bmatrix}.
\end{aligned}$$

Hence:

$$v(Y;\alpha) = \frac{L^{NN}(\alpha)V^I(Y;\alpha)-L^{YN}(\alpha)V^I(N;\alpha)}{L^{YY}(\alpha)L^{NN}(\alpha)-L^{YN}(\alpha)L^{NY}(\alpha)}, \tag{3}$$

$$v(N;\alpha) = \frac{L^{YY}(\alpha)V^I(N;\alpha)-L^{NY}(\alpha)V^I(Y;\alpha)}{L^{YY}(\alpha)L^{NN}(\alpha)-L^{YN}(\alpha)L^{NY}(\alpha)}. \tag{4}$$

In view of the assumptions made in section 1.7, the following hold:

$$\begin{aligned}
L^Y(Y,Y) &= \frac{1}{1-\delta} - \frac{1}{1-\delta^2} = \frac{\delta}{1-\delta^2},\\
L^N(Y,Y) &= 0,\\
L^Y(Y,N) &= \delta(1-\delta)\delta^{2T}[1+\delta^2+\dots]\frac{1}{1-\delta} = \frac{\delta\cdot\delta^{2T}}{1-\delta^2}\\
&= L^Y(N,Y) = L^Y(N,N)\\
L^N(Y,N) &= \delta(1-\delta)[1+\delta^2+\cdots+\delta^{2(T-1)}]\frac{1}{1-\delta}\\
&= \delta(1-\delta)\frac{1-\delta^{2T}}{1-\delta^2}\frac{1}{1-\delta} = \frac{\delta(1-\delta^{2T})}{1-\delta^2}\\
&= L^N(N,Y) = L^N(N,N).
\end{aligned}$$

It follows then:

$$\begin{aligned}
L^{YY}(\alpha) &= \frac{1}{1-\delta} - \alpha\frac{\delta}{1-\delta^2} - (1-\alpha)\frac{\delta\cdot\delta^{2T}}{1-\delta^2}\\
&= \frac{1+\delta-\alpha\delta-(1-\alpha)\delta\delta^{2T}}{1-\delta^2} = \frac{1+(1-\alpha)\delta(1-\delta^{2T})}{1-\delta^2}\\
L^{YN}(\alpha) &= -(1-\alpha)\frac{\delta(1-\delta^{2T})}{1-\delta^2}\\
L^{NY}(\alpha) &= -\alpha\frac{\delta\cdot\delta^{2T}}{1-\delta^2} - (1-\alpha)\frac{\delta\cdot\delta^{2T}}{1-\delta^2} = -\frac{\delta\cdot\delta^{2T}}{1-\delta^2}\\
L^{NN}(\alpha) &= \frac{1}{1-\delta} - \alpha\frac{\delta(1-\delta^{2T})}{1-\delta^2} - (1-\alpha)\frac{\delta(1-\delta^{2T})}{1-\delta^2} = \frac{1+\delta\cdot\delta^{2T}}{1-\delta^2}\\
V^I(Y;\alpha) &= \alpha\times\frac{1}{1-\delta^2}\times c + (1-\alpha)\times\frac{1}{1-\delta^2}\times[(1-\delta^{2T})d+\delta^{2T}c]\\
&= \frac{c-(1-\alpha)(1-\delta^{2T})(c-d)}{1-\delta^2}\\
V^I(N;\alpha) &= \alpha\times\frac{1}{1-\delta^2}\times[(1-\delta^{2T})d+\delta^{2T}c] + (1-\alpha)\times\frac{1}{1-\delta^2}\times[(1-\delta^{2T})d+\delta^{2T}c]\\
&= \frac{(1-\delta^{2T})d+\delta^{2T}c}{1-\delta^2} = \frac{c-(1-\delta^{2T})(c-d)}{1-\delta^2}.
\end{aligned}$$

Hence:

$$\begin{aligned}
&L^{YY}(\alpha)L^{NN}(\alpha) - L^{YN}(\alpha)L^{NY}(\alpha)\\
&\quad= \frac{[1+(1-\alpha)\delta(1-\delta^{2T})][1+\delta\delta^{2T}] - (1-\alpha)\delta(1-\delta^{2T})\cdot\delta\delta^{2T}}{(1-\delta^2)^2} \qquad (5)\\
&\quad= \frac{1+\delta\delta^{2T}+(1-\alpha)\delta(1-\delta^{2T})}{(1-\delta^2)^2}
\end{aligned}$$

$$
\begin{aligned}
&L^{NN}(\alpha)V^I(Y;\alpha) - L^{YN}(\alpha)V^I(N;\alpha)\\
&\quad = \frac{(1+\delta\delta^{2T})[c-(1-\alpha)(1-\delta^{2T})(c-d)] + (1-\alpha)\delta(1-\delta^{2T})[c-(1-\delta^{2T})(c-d)]}{(1-\delta^2)^2}\\
&\quad = \frac{[1+\delta\delta^{2T}+(1-\alpha)\delta(1-\delta^{2T})][c-(1-\delta^{2T})(c-d)]}{(1-\delta^2)^2}\\
&\qquad + \frac{(1+\delta\delta^{2T})\alpha(1-\delta^{2T})(c-d)}{(1-\delta^2)^2}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&L^{YY}(\alpha)V^I(N;\alpha) - L^{NY}(\alpha)V^I(Y;\alpha)\\
&\quad = \frac{[1+(1-\alpha)\delta(1-\delta^{2T})][c-(1-\delta^{2T})(c-d)] + \delta\delta^{2T}[c-(1-\alpha)(1-\delta^{2T})(c-d)]}{(1-\delta^2)^2}\\
&\quad = \frac{[1+\delta\delta^{2T}+(1-\alpha)\delta(1-\delta^{2T})][c-(1-\delta^{2T})(c-d)]}{(1-\delta^2)^2}\\
&\qquad + \frac{\alpha\delta\delta^{2T}(1-\delta^{2T})(c-d)}{(1-\delta^2)^2}
\end{aligned} \tag{7}
$$

In view of (5)-(7), (8) and (9) are rewrittern as:

$$
v(Y;\alpha) = c-(1-\delta^{2T})(c-d) + \frac{\alpha(1+\delta\delta^{2T})(1-\delta^{2T})(c-d)}{1+\delta\delta^{2T}+(1-\alpha)\delta(1-\delta^{2T})} \tag{8}
$$

$$
v(N;\alpha) = c-(1-\delta^{2T})(c-d) + \frac{\alpha\delta\delta^{2T}(1-\delta^{2T})(c-d)}{1+\delta\delta^{2T}+(1-\alpha)\delta(1-\delta^{2T})}. \tag{9}
$$

2.2.1 Composition of Matching Pool Poopulation

We next analyze the composition of matching pool population.

Let the matching pool population be of measure 1. Hence, there are Y players of measure α and N players of measure $1-\alpha$, creating $\frac{1}{2}$ measure of new partnerships. Thus, among newly created matchings,

- YY-partnership are of measure $\frac{1}{2}\alpha^2$, players in such partnership are of size α^2;
- YN-partnership are of measure $\frac{1}{2}\alpha(1-\alpha)$, players are of size $\alpha(1-\alpha)$;
- NY-partnership are of measure $\frac{1}{2}(1-\alpha)\alpha$, players are of size $(1-\alpha)\alpha$;
- NN-partnership are of measure $\frac{1}{2}(1-\alpha)^2$, players are of size $(1-\alpha)^2$.

Suppose further that the society has been in stationary state. Hence, in the past, proportion of Y and N players in the matching pool has been the same.

It follows that there are $\frac{1}{2}\alpha^2\delta^{2t}$ measure of YY-partnerships which enter into $t+1$-th period, $\frac{1}{2}\alpha(1-\alpha)\delta^{2t}$ measure of YN-partnerships which enter into $t+1$-th period, etc.

We now consider how parnership of different types and of different ages break up in each period and how does each break up gives rise to players in the matching pool with or without reference letter.

YY-partnership For each $t = 0, 1, \dots$, there are $\frac{1}{2}\alpha^2\delta^{2t}$ measure of such partnerships (hence $\alpha^2\delta^2$ measure of players) with age $t+1$. Out of such partnerships:

- players of measure $\delta(1-\delta)\alpha^2\delta^{2t}$ lose their partner and return to the matching pool as Y status player,
- players of measure $(1-\delta)\alpha^2\delta^{2t}$ die and be replaced by newly born players who enter the matching pool as N status players.

Thus, summing for all $t = 0, 1, \dots$:

- Out of YY-partnership breakups, the size of Y status players who return to the matching pool is:

$$\delta(1-\delta)\alpha^2[1 + \delta^2 + \dots] = \frac{\delta(1-\delta)\alpha^2}{1-\delta^2} \Rightarrow Y\text{-player}.$$

- Out of YY-partnership breakups, the size of those who die (and being replaced by newly born who comes to the pool as N players) is:

$$\frac{(1-\delta)\alpha^2}{1-\delta^2} \Rightarrow N\text{-player}.$$

YN- and NY-partnership By a similar logic, out of YN-patnerships:

- Those return to MP as a Y player are of the size:

$$\frac{\delta(1-\delta)\alpha(1-\alpha)\delta^{2T}}{1-\delta^2} \Rightarrow Y\text{-player}.$$

- Those return to MP as a N playler are of the size:

$$\frac{\delta(1-\delta)\alpha(1-\alpha)(1-\delta^{2T})}{1-\delta^2} \Rightarrow N\text{-player}.$$

- Those who die and be replaced by newly born are:

$$\frac{(1-\delta)\alpha(1-\alpha)}{1-\delta^2} \Rightarrow N\text{-player}.$$

The case of NY-partnership is exacetly the same.

NN-partnership Out of NN-partnership:

- Those who return to MP as a Y player are of the size:

$$\frac{\delta(1-\delta)(1-\alpha)^2\delta^{2T}}{1-\delta^2} \Rightarrow Y\text{-plyaer}.$$

- Those who return to MP as a N player are of the size:

$$\frac{\delta(1-\delta)(1-\alpha^2)(1-\delta^{2T})}{1-\delta^2} \Rightarrow N\text{-player.}$$

- Those who die and be replaced by newly born are:

$$\frac{(1-\delta)(1-\alpha)^2}{1-\delta^2} \Rightarrow N\text{-player.}$$

Total Summing all:

- Those who appear as Y players are of the size:

$$\frac{\delta(1-\delta)}{1-\delta^2}[\alpha^2 + 2\alpha(1-\alpha)\delta^{2T} + (1-\alpha^2)\delta^{2T}] = \frac{\delta(1-\delta)[\alpha^2 + (1-\alpha^2)\delta^{2T}]}{1-\delta^2},$$

- Those who appear as N players are of the size:

$$\frac{\delta(1-\delta)}{1-\delta^2}[2\alpha(1-\alpha)(1-\delta^{2T}) + (1-\alpha)^2(1-\delta^{2T})] + \frac{1-\delta}{1-\delta^2}[\alpha^2 + 2\alpha(1-\alpha) + (1-\alpha)^2]$$
$$= \frac{1-\delta}{1-\delta^2} + \frac{\delta(1-\delta)(1-\alpha^2)}{1-\delta^2} + \frac{\delta(1-\delta)(1-\alpha^2)\delta^{2T}}{1-\delta^2}.$$

2.2.2 Stationary distribution

It follows that, if the supposed state is a stationary state, then:

$$\frac{\delta(1-\delta)[\alpha^2 + (1-\alpha^2)\delta^{2T}]}{1-\delta^2} = \alpha$$

or:

$$\delta\delta^{2T} - (1+\delta)\alpha + \delta(1-\delta^{2T})\alpha^2 = 0. \tag{10}$$

Let:

$$f(\alpha) = \delta\delta^{2T} - (1+\delta)\alpha + \delta(1-\delta^{2T})\alpha^2.$$

Then, straightforward compulations yield:

$$\begin{aligned} f(0) &= \delta\delta^{2T} > 0, & f(1) &= -1, \\ f'(0) &= -(1+\delta) < 0, & f'(1) &= -1 + \delta - 2\delta^{2T}. \end{aligned}$$

It follows that the smaller root of $f(\alpha)$, say $\alpha^*(\delta)$ must be in $(0,1)$, assuring the existence of desired stationary state.

2.2.3 BR condition in Cooperation Period

Consider a player who is in a cooperation period (i.e., either she is in a YY-partnership or she has spent at least T days within a partnership. Let $v_{RL}^{CP}(\alpha)$ be average value that she expects to obtain in the rest of her life.

By definition:

$$\begin{aligned} v_{RL}^{CP}(\alpha) &= \frac{\frac{1}{1-\delta^2}c + [\frac{1}{1-\delta} - \frac{1}{1-\delta^2}]v(Y;\alpha)}{\frac{1}{1-\delta}} \\ &= \frac{1}{1+\delta}c + \frac{\delta}{1+\delta}v(Y;\alpha). \end{aligned}$$

It follows that non-deviation is a best response during the coopertion period if:

$$\begin{aligned} g - c &\leq \frac{\delta^2}{1-\delta}[v_{RL}^{CP}(\alpha) - v(N;\alpha)] + \frac{\delta(1-\delta)}{1-\delta}[v(Y;\alpha) - v(N;\alpha)] \\ &= \frac{\delta^2}{1-\delta^2}[c - v(N;\alpha)] + \left\{\frac{\delta^3}{1-\delta^2} + \frac{\delta(1-\delta^2)}{1-\delta^2}\right\}[v(Y;\alpha) - v(N;\alpha)] \\ &= \frac{\delta^2}{1-\delta^2}[c - v(N;\alpha)] + \frac{\delta}{1-\delta^2}[v(Y;\alpha) - v(N;\alpha)]. \end{aligned} \tag{11}$$

Comparison with NRL model Compared with no reference letter (NRL) model, whose corresponding condition is identical with (2), the first term on the RHS is smaller,

$$c - v(N;\alpha) < c - v_{NRL}^M,$$

making the BR condintion more restrictive than the no reference letter (NRL) model, but the second term on the RHS is larger,

$$v(Y;\alpha) - v(N;\alpha) > v_{NRL}^M - v_{NRL}^M,$$

making the BR condition less restrictive than the NRL model. As the following straightforward computation shows, the latter effect dominates the former and trust capital is larger with reference letters.

In view of (9):

$$\begin{aligned} c - v(N;\alpha) &= \left[1 - \frac{\alpha\delta\delta^{2T}}{1+\delta\delta^{2T} + (1-\alpha\delta(1-\delta^{2T}))}\right](1-\delta^{2T})(c-d), \\ v(Y;\alpha) - v(N;\alpha) &= \frac{\alpha}{1+\delta\delta^{2T} + (1-\alpha)\delta(1-\delta^{2T})}(1-\delta^{2T})(c-d). \end{aligned}$$

It follows then that (11) is rewritten as:

$$\begin{aligned} g - c &\leq \frac{\delta^2}{1-\delta^2}\left[1 + \frac{\alpha(1-\delta^{2(T+1)})}{\delta[1+\delta\delta^{2T} + (1-\alpha\delta(1-\delta^{2T}))]}\right](1-\delta^{2T})(c-d) \\ &:= TC^{RL}(\alpha). \end{aligned} \tag{12}$$

With reference letter, trust capital is amplified by the factor of $\Gamma > 1$ where:

$$\begin{aligned} TC^{RL}(\alpha) &= \left[1 + \frac{\alpha(1-\delta^{2(T+1)})}{\delta[1+\delta^{2T}+(1-\alpha)\delta(1-\delta^{2T})]}\right] \times TC^{NRL} \\ &= \Gamma \times TC^{NRL}. \end{aligned} \tag{13}$$

2.2.4 BR condition with and without reference letter

So far, there are two factors that make model of reference letter different from that of no reference letter.

First, because players expect to receive reference letter when partnership breaks up in cooperation period, average payoff $v(N;\alpha)$ (even for a player who has no reference letter) is higher in the reference letter model than that in no-reference letter model v^M_{NRL}, i.e., $v(N;\alpha) > v^M_{NRL}$. This effect, which we might call **market value effect**, increases the average payoff in the reference letter model even when we keep the length of trust-building periods the same as in the no-reference letter model.

Second, the size of trust capital is different between the two models. As we saw in the previous subsection, there are two effects for the size of trust capital. First, because average payoff for those who are sanctioned is higher with RL, the effect of sanction for deviators during "the periods that non-deviator would have enjoyed the cooperation period with the current partner" (i,e., the first term of the RHS of (11) is smaller with RL. We may call it **within partnership sanction effect.**

However, with reference letter possibility, whether or not holding reference letter results in a difference in payoffs even after the relatioship with current partner is terminated. We may call it **social sanction effect.** As (13) shows, the social sanction effect dominates the within partnership sanction effect and the trust capital is always larger with RL than the model without it.

It follows that the cooperation period can start earlier with reference letter possibility than that without it. Namely, the minimum length of trust-building periods for c_T to be monomorphic NSD, $\underline{\tau}^{NRL}(\delta, G)$, for NRL model is longer than the minimum length for c_T to constitute a monomorphic NSD, $\underline{\tau}^{RL}(\delta, G)$, for RL model. This effect may be termed as **trust capital effect**. This creates further payoff improvement by the introduction of reference letters.

To summaraize:

Proposition 1.

- *Even with the same length of trustbuilding periods,*
  - *Expected average payoff for players newly arriving in the matching pool is larger with reference letter, i.e.,* $v^M_{NRL} < v(N;\alpha)$.
  - *Trust capital is larger with reference letter, or,* $TC^{RL} > TC^{NRL}$.
- *Minimum length of trustbuilding periods,* $\underline{\tau}(\delta, G)$*, is shorter with reference letter than without reference letter, or,* $\underline{\tau}^{RL}(\delta, G) < \underline{\tau}^{NRL}(\delta, G)$.