wLec 06, 2020 年 11 月 11 日演習問題解答

I 以下の積分の値を求めましょう.
(1) $\int_0^1 e^{-2t}dt$, **(2)** $\int_0^{\frac{\pi}{2}} \sin t dt$, **(3)** $\int_0^{\frac{\pi}{2}} \sin 2t dt$, **(4)** $\int_1^2 \frac{1}{x^3}dx$, **(5)** $\int_1^8 x^{\frac{1}{3}}dx$, **(6)** $\int_1^2 (x-1)^4 dx$,
(7) $\int_1^2 \frac{1}{2x+1}dx$

(1) $\left(e^{-2t}\right)' = -\frac{1}{2}e^{-2t}$ から $\left(-\frac{1}{2}e^{-2t}\right)' = e^{-2t}$ となるので

$$\int_0^1 e^{-2t}dt = \left[-\frac{1}{2}e^{-2t}\right]_0^1 = -\frac{1}{2}\left(e^{-2}-1\right) = \frac{e^2-1}{2e^2}$$

(2)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin t dt &= \int_0^{\frac{\pi}{2}} (-\cos t)' dt \\ &= [-\cos t]_0^{\frac{\pi}{2}} = -(0-1) = 1\end{aligned}$$

(3) $(\cos 2t)' = -2\sin 2t$ から $\left(-\frac{1}{2}\cos 2t\right)' = \sin 2t$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin 2t dt &= \left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2}\left(\cos\pi - \cos 0\right) = -\frac{1}{2}(-1-1) = 1\end{aligned}$$

(4) $\left(\frac{1}{x^2}\right)' = -\frac{2}{x^3}$ から $\left(-\frac{1}{2}\cdot\frac{1}{x^2}\right)' = \frac{1}{x^3}$

$$\int_1^2 \frac{1}{x^3}dx = \left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_1^2 = -\frac{1}{2}\left(\frac{1}{4}-1\right) = \frac{3}{8}$$

(5) $\left(x^{\frac{4}{3}}\right)' = x^{\frac{1}{3}}$ から $\left(\frac{3}{4}x^{\frac{4}{3}}\right)' = x^{\frac{1}{3}}$

$$\int_1^8 x^{\frac{1}{3}}dx = \left[\frac{3}{4}x^{\frac{4}{3}}\right]_1^8 = \frac{3}{4}\left(8\cdot 2-1\right) = \frac{45}{4}$$

(6) $\left\{(x-1)^5\right\}' = 5(x-1)^4$ から $\left\{\frac{1}{5}(x-1)^5\right\}' = (x-1)^4$

$$\int_1^2 (x-1)^4 dx = \left[\frac{1}{5}(x-1)^5\right]_1^2 = \frac{1}{5}$$

(7) $\{\log(2x+1)\}' = 2\cdot\frac{1}{2x+1}$ から $\left\{\frac{1}{2}\log(2x+1)\right\}' = \frac{1}{2x+1}$

$$\int_1^2 \frac{1}{2x+1}dx = \left[\frac{1}{2}\log(2x+1)\right]_1^2 = \frac{1}{2}(\log 5 - \log 3) = \frac{1}{2}\log\frac{5}{3}$$

II $\mathbf{R}^2$ の開集合 U 上で定義された関数 $f(x,y)$ が与えられているとします．他方 U の中に C^2 級の曲線 $(x(t),y(t))$ が与えられているとします．このとき

$$F(t) := f(x(t),y(t))$$

を定義します．このとき

$$F''(t) := \Big(H(f)(x(t),y(t))\left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right), \left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right)\Big) + \Big(\nabla(f)(x(t),y(t)), \left(\begin{smallmatrix} x''(t) \\ y''(t) \end{smallmatrix}\right)\Big)$$

が成立することを示しましょう．

解答

$$F'(t) = f_x((x(t),y(t)) \cdot x'(t) + f_y((x(t),y(t)) \cdot y'(t)$$

をもう一度 t で微分すると

$$\begin{aligned} F''(t) &= (f_{xx}((x(t),y(t)) \cdot x'(t) + f_{xy}((x(t),y(t)) \cdot y'(t)) \cdot x'(t) + f_x((x(t),y(t)) \cdot x''(t) \\ &\quad + (f_{yx}((x(t),y(t)) \cdot x'(t) + f_{yy}((x(t),y(t)) \cdot y'(t)) \cdot y'(t) + f_y((x(t),y(t)) \cdot y''(t) \\ &= f_{xx}((x(t),y(t)) \cdot (x'(t))^2 + 2f_{xy}((x(t),y(t)) \cdot x'(t)y'(t) + f_{yy}((x(t),y(t)) \cdot (y'(t))^2 \\ &\quad + f_x((x(t),y(t)) \cdot x''(t) + f_y((x(t),y(t)) \cdot y''(t) \\ &= \Big(\left(\begin{smallmatrix} f_{xx}((x(t),y(t)) & f_{xy}((x(t),y(t)) \\ f_{yx}((x(t),y(t)) & f_{yy}((x(t),y(t)) \end{smallmatrix}\right)\left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right), \left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right)\Big) + \Big(\left(\begin{smallmatrix} f_x((x(t),y(t)) \\ f_y((x(t),y(t)) \end{smallmatrix}\right), \left(\begin{smallmatrix} x''(t) \\ y''(t) \end{smallmatrix}\right)\Big) \\ &= \Big(H(f)(x(t),y(t))\left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right), \left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix}\right)\Big) + \Big(\nabla(f)(x(t),y(t)), \left(\begin{smallmatrix} x''(t) \\ y''(t) \end{smallmatrix}\right)\Big) \end{aligned}$$