Strictly Quasi-concave functions of 2 variables

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Qusai-concave functions – Definition

- Let $\mathbf{R}^2_{++}$ be defined by

$$\mathbf{R}^2_{++} := \{(x, y) \in \mathbf{R}^2; x, y > 0\}$$

- u be a function on $\mathbf{R}^2_{++}$:

$$u : \mathbf{R}^2_{++} \longrightarrow \mathbf{R}$$

- **Definition** u is called quasi-concave if

$$u(\mathrm{a}) \leq u(\mathrm{b}) \Rightarrow u(\mathrm{a}) \leq u((1 - t)\mathrm{a} + t\mathrm{b}) \ (0 \leq t \leq 1)$$

Quasi-concave functions – Criterion

- **Theorem** u is quasi-concave if and only if

$$U_c := \{(x, y) \in \mathbf{R}^2_{++};\ u(x, y) \geq c\}$$

is convex for any $x \in \mathbf{R}$.

- **(proof)** We assume u to be quasi-concave. We take two points $\mathbf{a}, \mathbf{b} \in U_c$ satisfying

$$c \leq u(\mathbf{a}) \leq u(\mathbf{b}).$$

Then $c \leq u(\mathbf{a}) \leq u((1-t)\mathbf{a} + t\mathbf{b})$ for any $t \in [0, 1]$. This means $\overline{\mathbf{ab}} \subset U_c$.

Quasi-concave functions – Criterion(2)

- **(Proof (continued))** Conversely we assume that

$$\{(x, y) \in \mathbf{R}^2_{++};\ u(x, y) \geq c\}$$

  is convex for any $x \in \mathbf{R}$.
- Take any two points $\mathbf{a}, \mathbf{b} \in \mathbf{R}^2_{++}$. We may assume $u(\mathbf{a}) \leq u(\mathbf{b})$.
  We make use of the fact that

$$\{(x, y) \in \mathbf{R}^2_{++};\ u(x, y) \geq u(\mathbf{a})\}$$

  is convex to deduce

$$u(\mathbf{a}) \leq u((1 - t)\mathbf{a} + t\mathbf{b})\ (0 \leq t \leq 1).$$

Concave functions are quasi-concave

- **Theorem** If u is concave,

$$U_c := \{(x, y) \in \mathbf{R}^2_{++};\ u(x, y) \geq c\}$$

is convex for any $c \in \mathbf{R}$. Accordingly u is quasi-concave.

- **(proof)** Take any two points $\mathrm{a}, \mathrm{b} \in U_c$. It follows from the concavity of u that

$$u((1-t)\mathrm{a} + t\mathrm{b}) \geq (1-t)u(\mathrm{a}) + tu(\mathrm{b})\ (0 \leq t \leq 1).$$

If $c \leq u(\mathrm{a}) \leq u(\mathrm{b})$,

$$(1-t)u(\mathrm{a}) + tu(\mathrm{b}) \geq u(\mathrm{a}) \geq c.$$

Thus $(1-t)\mathrm{a} + t\mathrm{b} \in U_c$ for any $t \in [0, 1]$

Stricltly Qusai-concave fcts – Definition

- **Definition** u is called stricltly quasi-concave if

  $$u(\mathbf{a}) \leq u(\mathbf{b}) \Rightarrow u(\mathbf{a}) < u((1-t)\mathbf{a} + t\mathbf{b}) \ (0 < t < 1)$$

  for any two distinct $\mathbf{a}, \mathbf{b} \in \mathbf{R}^2_{++}$.
- If u is stricltly concave, then u is stricltly quasi-concave.

SQC fcts – Sufficient condition

- **Main Theorem** We assume that u is of C^2 class. The u is stricltly quasi-concave if

$$\begin{vmatrix} 0 & u_x \\ u_x & u_{xx} \end{vmatrix} < 0, \begin{vmatrix} 0 & u_x & u_y \\ u_x & u_{xx} & u_{xy} \\ u_y & u_{yx} & u_{yy} \end{vmatrix} > 0.$$

- If $u_x(\mathbf{a}) > 0$ at any $\mathbf{a} \in \mathbf{R}^2_{++}$ and if $(H(u)(\mathbf{a})\vec{v}, \vec{v}) < 0$ for any non-zero $\vec{v} \in \mathbf{R}^2$ and at any point $\mathbf{a} \in \mathbf{R}^2_{++}$, the above condition is satisfied.

What Sufficient condition Menas

- We assume $u_x(\mathrm{a}) > 0$ and

$$\begin{vmatrix} 0 & u_x & u_y \\ u_x & u_{xx} & u_{xy} \\ u_y & u_{yx} & u_{yy} \end{vmatrix} > 0 \text{ at } \mathbf{a}.$$

- We apply Implicit function theorem to find a funxction $x = \varphi(y)$ with the property that

$$\begin{aligned} & \{(x, y) \in B_\delta(\mathrm{a}); u(x, y) \geq u(\mathrm{a})\} \\ = \ & \{(x, y) \in B_\delta(\mathrm{a}); x \geq \varphi(y)\} \end{aligned}$$

for sime $\delta > 0$.

What Sufficient condition Menas(2)

- Moreover we have

$$\varphi''(a_2) = \frac{1}{u_x(\mathrm{a})^3} \begin{vmatrix} 0 & u_x(\mathrm{a}) & u_y(\mathrm{a}) \\ u_x(\mathrm{a}) & u_{xx}(\mathrm{a}) & u_{xy}(\mathrm{a}) \\ u_y(\mathrm{a}) & u_{yx}(\mathrm{a}) & u_{yy}(\mathrm{a}) \end{vmatrix} > 0$$

- Thus $\varphi''(y) > 0$ in a neighborhood of $a_?$. This means that if we take another small $\delta > 0$

$$(x, y) \in B_\delta(\mathrm{a}) \cap Z_-, (x, y) \neq \mathrm{a} \Rightarrow u(x, y) < u(\mathrm{a})$$

where
$Z_- := \{x \in \mathrm{R}^2_{++}; \nabla(u)(\mathrm{a}) \cdot (x - a_1, y - a_2) \leq 0\}$.

Proof of the Main Theorem

- **(2nd step)** The contraposition of the previous statement is as follows:
  For any $\mathbf{a} \in \mathbf{R}^2_{++}$, we can find $\delta > \mathbf{0}$ satisfying the condition that

$$(x, y) \in B_\delta(\mathrm{a}), u(\mathrm{a}) \leq u(x, y), (x, y) \neq \mathrm{a}$$
$$\Rightarrow \quad \nabla(u)(\mathrm{a}) \cdot (x - a_1, y - a_2) > \mathbf{0}.$$

Proof of the Main Theorem(2)

- We take two distinct points $\mathbf{a}_0, \mathbf{a}_1$ with $u(\mathbf{a}_0) \leq u(\mathbf{a}_1)$. We define $\mathbf{a}_t = (1-t)\mathbf{a}_0 + t\mathbf{a}_1$ and

$$U(t) := u(\mathbf{a}_t).$$

  Moreover we assume that $U(t^*) = \min_{0 \leq t \leq 1} U(t)$.
- since $U(0) \leq U(1)$, we may assume $0 \leq t^* < 1$. We shall show $t^* = 0$ by proof by contradiction. We assume that $0 < t^* < 1$.
- We have $U(t^*) \leq U(t)$ for any $t \in [0,1]$, which means that

$$u(\mathbf{a}_{t^*}) \leq u(\mathbf{a}_t) \ (t \in [0,1]).$$

Proof of the Main Theorem(3)

- We make use of the statement of the 2nd step at $\mathbf{aF_{t^*}}$.Then it follows that

$$\nabla(u)(\mathbf{a}_{t^*}) \cdot (\mathbf{a_t} - \mathbf{a_{t^*}}) > 0$$

$$(t - t^*)\nabla(u)(\mathbf{a}_{t^*}) \cdot (\mathbf{a}_1 - \mathbf{a}_0)$$

for $t \equiv t^*$ and $t \neq t^*$. If $0 < t^* < 1$, the value of $t - t^*$ takes the both signs, it is impossible. Thus we have proved $t^* = 0$. Moreover we have shown that $U(0) < U(t)$ for $t \in (0, 1)$.